Chapter 9
Method of Weighted Residuals
9-1 Introduction

Method of Weighted Residuals (MWR) is an approximate technique for solving boundary value problems.

It utilizes a trial functions satisfying the prescribed boundary conditions, and integral formulation to minimize the error, over the problem domain.

The general concept is described for a one-dimensional case. But the concept can easily be extended to two-dimensional and three-dimensional cases.
9-2 General Concept

Given a differential equation of the general form,

\[ D[y(x), x] = 0 \quad a < x < b \]  \hspace{1cm} (i)

subject to **homogeneous** boundary conditions

\[ y(a) = y(b) = 0 \]  \hspace{1cm} (ii)

The method of weighted residuals seeks an **approximate** solution in the form

\[ y^*(x) = \sum_{i=1}^{n} c_i N_i(x) \]  \hspace{1cm} (iii)

where \( y^* \) is the approximate solution expressed as the product of \( c_i \) unknown (i.e. constant parameters to be determined), and \( N_i(x) \) are trial functions.

**Note**: The solution represented in Eq.(iii) is not an exact one!
When the assumed solution of eq.(iii) is substituted into the differential equation of eq.(i), a residual error $R(x)$ (residual) will result, which is given by

$$R(x) = D\left[ y^*(x), \ x \right] \neq 0 \quad \text{(iv)}$$

**Note:** Residual $R(x)$ is also a function of the unknown parameters, $c_i$.

The method of weighted residuals (MWR) requires that the unknown parameters $c_i$ be evaluated such that,

$$\int_a^b w_i(x)R(x)dx = 0 \quad i = 1, n \quad \text{(v)}$$

Where $w_i(x)$ represents $n$ arbitrary weighting functions.

**Note:** On integration, Eq.(v) results in $n$ algebraic equations, which can be solved for the $n$ values of $c_i$.  

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Eq. (v) expresses that the sum (integral) of the weighted residual error over the domain of the problem is zero.

The solution is exact at the end points (the boundary conditions must be satisfied) but, in general, at any interior point the residual error is nonzero.

**Note:** Several variations of the MWR exist and the technique vary primarily on how the weighting factors are selected. The most common techniques are point collocation, sub-domain collocation, least squares and the Galerkin’s method.

We will only discuss the Galerkin’s method as it is quite simple to use and readily adaptable to the finite element method.
9-3 Galerkin’s Method

In Galerkin’s weighted residual method, the weighting functions are chosen to be identical to the trial functions, i.e.

\[ w_i(x) = N_i(x) \quad i = 1, n \]

Therefore, the unknown parameters are determined via

\[ \int_a^b w_i(x)R(x)dx = \int_a^b N_i(x)R(x)dx = 0 \quad i = 1, n \]

Again, the above integration results in \( n \) algebraic equations for evaluation of the \( n \) unknown parameters.
Example 9-1

Use Galerkin’s weighted residuals to obtain an approximate solution for the differential equation,

\[
\frac{d^2 y}{dx^2} - 10x^2 = 5 \quad 0 \leq x \leq 1
\]

with boundary conditions \( y(0) = y(1) = 0 \)

Solution

Since the integral equation have quadratic terms, the suitable trial functions will be polynomial. For homogeneous boundary conditions at \( x = a \) and \( x = b \), the general form is

\[
N(x) = (x - x_a)^p (x - x_b)^q
\]

where \( p \) and \( q \) are positive integers, that satisfy the boundary conditions. Using a single trial function, the simplest form will be

\[
N_1(x) = x(x - 1)
\]
Using this trial function, the approximate solution is

\[ y^* (x) = c_1 x(x - 1) \]

and the first and second derivatives are

\[ \frac{dy^*}{dx} = c_1 (2x - 1) \]

\[ \frac{d^2 y^*}{dx^2} = 2c_1 \]

**Note:** The selected trial function does not satisfy the physics of the problem because the second derivative is a constant, whereas according to the original differential equation, the second derivative must be a quadratic.

Nevertheless, we will continue with the example to illustrate the procedure.
Substitution of the second derivative of $y^*(x)$ into the differential equation yields the residual,

$$R(x; c_1) = 2c_1 - 10x^2 - 5$$

which is clearly a nonzero value.

Substituting the residual into the integral equation we get

$$\int_0^1 N(x)R(x; c_1)dx = 0$$

$$\int_0^1 x(x-1)(2c_1 - 10x^2 - 5)dx = 0$$

which, after integration yields

$$c_1 = 4$$

So the approximate solution is obtained as

$$y^*(x) = 4x(x - 1)$$
For this example, the exact solution can be obtained by integrating the differential equation twice, i.e.

\[\frac{dy}{dx} = \int d^2y \, dx = \int \left(10x^2 + 5\right) \, dx = \frac{10x^3}{3} + 5x + C_1\]

\[y(x) = \int dy \, dx = \int \left(\frac{10x^3}{3} + 5x + C_1\right) \, dx = \frac{5x^4}{6} + \frac{5x^2}{2} + C_1x + C_2\]

Applying the boundary conditions i.e. \(y(0) = 0\) and \(y(1) = 0\), we get

\[C_2 = 0 \quad \text{and} \quad C_1 = -\frac{10}{3}\]

Therefore the exact solution is

\[y(x) = \frac{5}{6} x^4 + \frac{5}{2} x^2 - \frac{10}{3} x\]
A graphical comparison between the two solutions is shown below. The approximate solution agrees reasonably well with the exact solution.
Example 9-2

Obtain two-term Galerkin’s solution for

\[
\frac{d^2 y}{dx^2} - 10x^2 = 5 \quad 0 \leq x \leq 1
\]  

(i)

with boundary conditions \( y(0) = y(1) = 0 \) and using the following trial functions

\[ N_1(x) = x(x-1); \quad N_2(x) = x^2(x-1) \]

Solution

Using these trial functions, the approximate solution is

\[ y^*(x) = c_1x(x-1) + c_2x^2(x-1) \]  

(ii)

The second derivatives of the above is

\[
\frac{d^2 y^*}{dx^2} = 2c_1 + 2c_2(3x-1)
\]  

(iii)
Substituting the second derivative of $y^*(x)$ into the differential equation yields the residual

$$R(x; c_1; c_2) = 2c_1 + 2c_2(3x - 1) - 10x^2 - 5$$

Substituting the residual into the integral equation we get

$$\int_0^1 x(x - 1)[2c_1 + 2c_2(3x - 1) - 10x^2 - 5]dx = 0$$

$$\int_0^1 x^2(x - 1)[2c_1 + 2c_2(3x - 1) - 10x^2 - 5]dx = 0$$

which, after integration and solving the equations yields

$$c_1 = \frac{19}{6}; \quad c_2 = \frac{5}{3}$$
So the two-term approximation solution is
\[
y^\star(x) = \frac{19}{6} x (x - 1) + \frac{5}{3} x^2 (x - 1)
\]
\[
y^\star(x) = \frac{5}{3} x^3 + \frac{3}{2} x^2 - \frac{19}{6} x
\]

Recall, the one-term approximation solution is
\[
y^\star(x) = 4x(x - 1)
\]

and the exact solution is
\[
y(x) = \frac{5}{6} x^4 + \frac{5}{2} x^2 - \frac{10}{3} x
\]
A graphical comparison between the three solutions is shown. We see that the two-term approximate solution matches quite closely to the exact solution.
9-4 The Galerkin’s Finite Element Method

Element Formulation

Consider a problem represented by a differential equation

\[
\frac{d^2 y}{dx^2} + f(x) = 0; \quad x_j \leq x \leq x_{j+1}
\]

subjected to boundary conditions

\[
y(x_j) = y_j; \quad y(x_{j+1}) = y_{j+1}
\]

The problem domain is represented by
The trial or interpolation functions are chosen such that

\[ N_1(x) = \frac{x_{j+1} - x}{x_{j+1} - x_j}, \quad x_j \leq x \leq x_{j+1} \quad (i) \]

\[ N_2(x) = \frac{x - x_j}{x_{j+1} - x_j} \]

At the boundaries of the element domain, these trial functions have the values

\[ N_1(x_j) = 1; \quad N_1(x_{j+1}) = 0 \]

\[ N_2(x_j) = 0; \quad N_2(x_{j+1}) = 1 \quad (ii) \]

Using the above trial functions, the approximate solution will be

\[ y^{(e)}(x) = y_jN_1(x) + y_{j+1}N_2(x) \quad (iii) \]
Substituting the approximate solution into the differential equation yields the residual, given by

\[
R^{(e)} = \frac{d^2 y^{(e)}}{dx^2} + f(x)
\]

\[
R^{(e)} = \frac{d^2}{dx^2} \left[ y_j N_1(x) + y_{j+1} N_2(x) \right] + f(x)
\]  \hspace{1cm} (iv)

Applying the Galerkin’s weighted residual criterion results in

\[
\int_{x_j}^{x_{j+1}} N_i(x) R^{(e)} dx = 0 \hspace{1cm} (i = 1, 2)
\]

\[
\int_{x_j}^{x_{j+1}} N_i(x) \cdot \frac{d^2 y^{(e)}}{dx^2} \cdot dx + \int_{x_j}^{x_{j+1}} N_i(x) \cdot f(x) \cdot dx = 0
\]  \hspace{1cm} (v)

Applying integration by part on the first integral term, we get

Next page...
\[ N_i(x) \frac{dy^{(e)}}{dx} \bigg|_{x_j}^{x_{j+1}} - \int_{x_j}^{x_{j+1}} \frac{dN_i}{dx} \cdot \frac{dy^{(e)}}{dx} \, dx \]

\[ + \int_{x_j}^{x_{j+1}} N_i(x) \cdot f(x) \, dx = 0 \quad (i = 1, 2) \quad \text{(vi)} \]

Evaluate the non-integral terms at the boundaries and rearranging the equations we get

\[ \int_{x_j}^{x_{j+1}} \frac{dN_1}{dx} \cdot \frac{dy^{(e)}}{dx} \, dx = \int_{x_j}^{x_{j+1}} N_1(x) \cdot f(x) \, dx + \frac{dy^{(e)}}{dx} \bigg|_{x_j}^{x_{j+1}} \quad \text{(vii)} \]

\[ \int_{x_j}^{x_{j+1}} \frac{dN_2}{dx} \cdot \frac{dy^{(e)}}{dx} \, dx = \int_{x_j}^{x_{j+1}} N_2(x) \cdot f(x) \, dx + \frac{dy^{(e)}}{dx} \bigg|_{x_j}^{x_{j+1}} \]

Setting \( j = 1 \), and substitute the approximate solution \( y^{(e)}(x) \) into the above eq.(vii), we get
\[
\int_{x_1}^{x_2} \frac{dN_1}{dx} \cdot \left[ \begin{array}{c}
y_1 \\
y_2
\end{array} \right] dx = \int_{x_1}^{x_2} N_1(x) \cdot f(x)dx + \left. \frac{dy^{(e)}}{dx} \right|_{x_1} \\
\int_{x_1}^{x_2} \frac{dN_2}{dx} \cdot \left[ \begin{array}{c}
y_1 \\
y_2
\end{array} \right] dx = \int_{x_1}^{x_2} N_2(x) \cdot f(x)dx + \left. \frac{dy^{(e)}}{dx} \right|_{x_2}
\]

which can be written in the matrix form as

\[
\begin{bmatrix}
k_{11} & k_{12} \\
k_{21} & k_{22}
\end{bmatrix} \left\{ \begin{array}{c}
y_1 \\
y_2
\end{array} \right\} = \left\{ \begin{array}{c}
f_1 \\
f_2
\end{array} \right\}
\]

where,

\[
k_{ij} = \int_{x_1}^{x_2} \frac{dN_i}{dx} \cdot \frac{dN_j}{dx} \cdot dx \quad (i, j = 1, 2)
\]

**Note:** The RHS of eq. (viii) represents the forces acting at the two ends of the elements’ domain.
Assembly of Elements

Since the finite element solution is not exact, then if two elements are connected at a node, we have

\[ y^{(3)} (x_4) = y^{(4)} (x_4) \]

\[ \frac{dy^{(3)}}{dx} \bigg|_{x_4} \neq \frac{dy^{(4)}}{dx} \bigg|_{x_4} \]

For exact solution,

\[ \frac{dy^{(3)}}{dx} \bigg|_{x_4} = \frac{dy^{(4)}}{dx} \bigg|_{x_4} \]
9-5 Application of Galerkin’s FE Method

1. One-Dimensional Problem

Consider a prismatic bar under an axial loading. The differential equation governing this problem is given by

\[ \frac{d\sigma_x}{dx} = \frac{d}{dx} (E \varepsilon_x) = E \cdot \frac{d^2u(x)}{dx^2} = 0 \]  

(i)

The approximate solution for displacement \( u(x) \) is

\[ u(x)^* = u_1 N_1(x) + u_2 N_2(x) \]  

(ii)

\[ \begin{array}{c}
1 \\
\bullet \\
u_1 \\
\text{e}
\end{array} \quad \begin{array}{c}
2 \\
\bullet \\
u_2 \\
x
\end{array} \quad \begin{array}{c}
\text{e} \\
L
\end{array} \]

Note: The domain of our solution is the volume of the element.
The trial functions used in Eq. (ii) are chosen such that

\[ N_1(x) = \left(1 - \frac{x}{L}\right); \quad N_2 = \left(\frac{x}{L}\right) \]  

(iii)

Hence, the approximate solution can be written as

\[ u(x)^* = u_1\left(1 - \frac{x}{L}\right) + u_2\left(\frac{x}{L}\right) \]  

(iv)

Substituting the approximate solution of Eq. (iv) into the differential equation of Eq. (i) results in a residual given by

\[ R^{(e)} = E \cdot \frac{d^2 u(x)^*}{dx^2} = E \cdot \frac{d^2}{dx^2} \left[u_1\left(1 - \frac{x}{L}\right) + u_2\left(\frac{x}{L}\right)\right] \]  

(v)

Applying the Galerkin’s weighted residual criterion results in

\[ \iiint_V N_i(x) \cdot E \frac{d^2 u(x)^*}{dx^2} dV = 0 \]  

(vi)
Since \( dV = A \, dx \), where \( A \) is cross-sectional area of the bar, which is uniform, Eq.(iv) can then be expressed as

\[
\int_0^L N_i(x) \cdot E \frac{d^2 u(x)^*}{dx^2} \cdot A \cdot dL = 0
\]  
(vii)

Integrating Eq.(vii) by part and rearranging the result, we get

\[
AE \int_0^L \frac{dN_i}{dx} \cdot \frac{du(x)^*}{dx} \, dx = \left[ N_i(x)EA \cdot \frac{du(x)^*}{dx} \right]_0^L
\]  
(viii)

Substituting the approximate solution of Eq.(iv) into Eq.(viii) and solving the RHS of the equation, we obtain

\[
AE \int_0^L \frac{dN_1}{dx} \cdot \frac{d}{dx}(u_1N_1 + u_2N_2) \, dx = -A\sigma \bigg|_{x=0}
\]  
(ix)

\[
AE \int_0^L \frac{dN_2}{dx} \cdot \frac{d}{dx}(u_1N_1 + u_2N_2) \, dx = -A\sigma \bigg|_{x=L}
\]
Combining Eq. (ix) and write in matrix form, we get

\[ AE \int_0^L \begin{bmatrix} \frac{dN_1}{dx} \frac{dN_1}{dx} & \frac{dN_1}{dx} \frac{dN_2}{dx} \\ \frac{dN_1}{dx} \frac{dN_2}{dx} & \frac{dN_2}{dx} \frac{dN_2}{dx} \end{bmatrix} \cdot dx \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \]  

(x)

Integrating individual term within the square bracket independently yields,

\[ \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \]  

(xi)

which is the system of linear equations for a 1-D bar element.

**Note:** The square matrix represents the stiffness matrix for the element.
2. Beam in Bending

Consider a portion of beam loaded with uniformly distributed load \( q(x) \) as shown.

The differential equation for this problem is represented by

\[
\frac{d^2}{dx^2} \left( EI_z \frac{d^2 v}{dx^2} \right) = q(x)
\]

Galerkin’s finite element method is applied by using an approximate solution for \( v(x) \) in the form

\[
v(x) = N_1(x) v_1 + N_2(x) \theta_1 + N_3(x) v_2 + N_4(x) \theta_2 = \sum_{i=1}^{4} N_i(x) \delta_i
\]
Substituting the approximate solution into the original differential equation yields the residuals, i.e.

\[
\int_{x_1}^{x_2} N_i(x) \left[ \frac{d^2}{dx^2} \left( EI_z \frac{d^2v}{dx^2} \right) - q(x) \right] \, dx = 0 \quad i = 1, 4
\]

Integrating the derivative terms by parts and assuming a constant \( EI_z \), we obtain

\[
N_i(x) EI_z \frac{d^3v}{dx^3} \bigg|_{x_1}^{x_2} - EI_z \int_{x_1}^{x_2} \frac{dN_i}{dx} \frac{d^3v}{dx^3} \, dx - \int_{x_1}^{x_2} N_i q(x) \, dx = 0 \quad i = 1, 4
\]

We observe that the first term of the above equation represents the shear force conditions at the element nodes, i.e.

\[
V = - \frac{dM}{dx} = - \frac{d}{dx} \left( EI_z \frac{d^2v}{dx^2} \right) = - EI_z \frac{d^3v}{dx^3}
\]
Integrating again by part and rearranging gives,

\[ EI_z \int_{x_1}^{x_2} \frac{d^2 N_i}{dx^2} \frac{d^2 v}{dx^2} \, dx = \int_{x_1}^{x_2} N_i q(x) \, dx - N_i EI_z \frac{d^3 v}{dx^3} \bigg|_{x_1}^{x_2} \]

\[ + \frac{dN_i}{dx} EI_z \frac{d^2 v}{dx^2} \bigg|_{x_1}^{x_2} \quad i = 1, 4 \]

This equation can be written in the matrix form,

\[ [k]^{(e)} \{\delta\} = \{F\} \]

where the terms of the stiffness matrix are defined by

\[ k_{ij} = EI_z \int_{x_1}^{x_2} \frac{d^2 N_i}{dx^2} \frac{d^2 N_j}{dx^2} \, dx \quad i, j = 1, 4 \]
The terms for the element force vector are defined by

\[ F_i = \int_{x_1}^{x_2} N_i q(x) \, dx + N_i V(x)|_{x_1}^{x_2} + \frac{dN_i}{dx} M(x)|_{x_1}^{x_2} \quad i = 1, 4 \]

Where the integral term represents the equivalent nodal forces and moments produced by the distributed load.

If \( q(x) = q = \text{constant (positive upward)} \), then substitution of the interpolation function into the above equation yields the element nodal force vector given by…

\[ \{F\} = \begin{pmatrix} \frac{qL}{2} - V_1 \\ \frac{qL^2}{12} - M_1 \\ \frac{qL}{2} + V_2 \\ -\frac{qL^2}{12} + M_2 \end{pmatrix} \]
3. One-Dimensional Heat Conduction

Application of the Galerkin’s finite element method to the problem of one-dimensional, steady-state heat conduction is developed based on the condition depicted in figure below.

\[ q_x(x) + \frac{dq_x}{dx} dx \]

**Note:** Surfaces of the body normal to the x-axis are assumed to be perfectly insulated.
Performing an energy balance across a small cubic element, we obtain the differential equation governing the steady-state heat conduction through the body, given by

\[ k_x \frac{d^2 T}{dx^2} + Q = 0 \]

where \( Q \) is an internal heat generation rate in \( W/m^3 \).

The approximate solution for the temperature distribution \( T^*(x) \) in the element is expressed as,

\[ T(x) = N_1(x)T_1 + N_2(x)T_2 \]

where \( T_1 \) and \( T_2 \) are the temperatures at nodes 1 and 2. The linear interpolation functions \( N_1 \) and \( N_2 \) are given by

\[ N_1(x) = \frac{x_{j+1} - x}{x_{j+1} - x_j}; \quad N_2(x) = \frac{x - x_j}{x_{j+1} - x_j} \]
Substituting the approximate solution into the original differential equation yields the residuals, i.e.

\[
\int_{x_1}^{x_2} \left( k_x \frac{d^2 T}{dx^2} + Q \right) N_i(x) A \ dx = 0 \quad i = 1, 2
\]

Integrating the first terms by parts, we obtain

\[
k_x A N_i(x) \left. \frac{dT}{dx} \right|_{x_1}^{x_2} - k_x A \int_{x_1}^{x_2} \frac{dN_i}{dx} \frac{dT}{dx} \ dx + A \int_{x_1}^{x_2} Q N_i(x) \ dx = 0
\]

Evaluating the first term of the above equation, substituting the approximate solution \( T^*(x) \) into the second term, and rearranging the results, we obtain two equations which can be expressed as

(see next page...
These equations can be written in a condensed matrix form as

\[
\begin{align*}
  k_x A \int_{x_1}^{x_2} \frac{dN_1}{dx} \left( \frac{dN_1}{dx} T_1 + \frac{dN_2}{dx} T_2 \right) dx &= A \int_{x_1}^{x_2} Q N_1 dx - k_x A \frac{dT}{dx} \bigg|_{x_1} \\
  k_x A \int_{x_1}^{x_2} \frac{dN_2}{dx} \left( \frac{dN_1}{dx} T_1 + \frac{dN_2}{dx} T_2 \right) dx &= A \int_{x_1}^{x_2} Q N_2 dx + k_x A \frac{dT}{dx} \bigg|_{x_2}
\end{align*}
\]

These equations can be written in a condensed matrix form as

\[
[k_T]^{(e)} \{T\} = \{f_Q\} + \{f_g\} \quad \ldots (i)
\]

where \([k_T]^{(e)}\) is the element conductivity matrix. The elements of this matrix are defined by

\[
k_{lm} = k_x A \int_{x_1}^{x_2} \frac{dN_l}{dx} \frac{dN_m}{dx} dx \quad l, m = 1, 2 \quad \ldots (ii)
\]
The first RHS term of eq.(i) is nodal “force” vector arising from internal heat generation with values defined by

\[ f_{Q1} = A \int_{x_1}^{x_2} Q N_1 \, dx \]

\[ f_{Q2} = A \int_{x_1}^{x_2} Q N_2 \, dx \]

and vector \( \{f_g\} \) represents the gradient boundary condition at the element nodes.

Performing the integrations indicated in eq.(ii) gives the element conductivity matrix as

\[
[k] = \frac{k_x A}{L} \begin{bmatrix}
1 & -1 \\
-1 & 1
\end{bmatrix}
\]
For constant internal heat generation $Q$, eq.(iii) results in the nodal vector,

$$\{f_Q\} = \begin{bmatrix} \frac{QAL}{2} \\ \frac{QAL}{2} \end{bmatrix}$$

The element gradient boundary condition $\{f_g\}$ is described by

$$\{f_g\} = k_x A \begin{bmatrix} -\frac{dT}{dx} \bigg|_{x_1} \\ \frac{dT}{dx} \bigg|_{x_2} \end{bmatrix} = A \begin{bmatrix} q \bigg|_{x_1} \\ -q \bigg|_{x_2} \end{bmatrix}$$
End of Chapter 9