A - INTRODUCTION AND OVERVIEW
Course Content:

A – INTRODUCTION AND OVERVIEW
Numerical method and Computer-Aided Engineering; Physical problems; Mathematical models; Finite element method;

B – REVIEW OF 1-D FORMULATIONS
Elements and nodes, natural coordinates, interpolation function, bar elements, constitutive equations, stiffness matrix, boundary conditions, applied loads, theory of minimum potential energy; Plane truss elements; Examples.

C – PLANE ELASTICITY PROBLEM FORMULATIONS
Constant-strain triangular (CST) elements; Plane stress, plane strain; Axisymmetric elements; Stress calculations; Programming structure; Numerical examples.
COMPUTER-AIDED ENGINEERING (CAE)

The use of computers to analyze and simulate the function (structural, motion, etc.) of mechanical, electronic or electromechanical systems.

• Computer-Aided Design (CAD)
  - Drafting
  - Solid modeling
  - Animation & visualization
  - Dimensioning & tolerancing

• Engineering Analyses
  - Analytical & numerical methods
The Process of FE analysis

PHYSICAL PROBLEM AND MATHEMATICAL MODEL

Buckling of Euler Column

\[ \frac{d^2 y}{dx^2} + \frac{P}{EI} y = 0 \]

Eigenvalue problem

\[
\begin{bmatrix}
  k_1 + k_2 & -k_2 \\
  -k_2 & k_2
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2
\end{bmatrix}
= \omega^2
\begin{bmatrix}
  m_1 & 0 \\
  0 & m_2
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2
\end{bmatrix}
\]
EXAMPLE OF A MODEL

Steady-state heat conduction through a thick wall

\[
\frac{d}{dx} \left( k \frac{dT}{dx} \right) + Q = 0
\]

Solution: \( T(x) \)

\[
\begin{bmatrix}
    k_{11} & k_{12} & \cdots & k_{1L} \\
    k_{21} & k_{22} & \cdots & k_{2L} \\
    \vdots & \vdots & \ddots & \vdots \\
    k_{L1} & k_{L2} & \cdots & k_{LL} + h
\end{bmatrix}
\begin{bmatrix}
    T_1 \\
    T_2 \\
    \vdots \\
    T_L
\end{bmatrix}
= 
\begin{bmatrix}
    R_1 \\
    R_2 \\
    \vdots \\
    R_L + hT_\infty
\end{bmatrix}
\]
REVIEW OF MATRIX ALGEBRA
Matrix Algebra

In this course, we need to solve system of linear equations in the form

\[ a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \]
\[ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \]
\[ \vdots \]
\[ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n \]  \hspace{1cm} (2-1)

where \( x_1, x_2, \ldots, x_n \) are the unknowns.

Eqn. (2-1) can be written in a matrix form as

\[ [A]\{x\} = \{b\} \]  \hspace{1cm} (2-2)

where \([A]\) is a \((n \times n)\) square matrix, \([x]\) and \([b]\) are \((n \times 1)\) vectors.
The square matrix \([A]\) and the \([x]\) and \([b]\) vectors are is given by,

\[
[A] = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{bmatrix}, \quad \{x\} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \text{and} \quad \{b\} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}
\]

\[ \vdots \quad (2-3) \]

**Note:**

Element located at \(i^{th}\) row and \(j^{th}\) column of matrix \([A]\) is denoted by \(a_{ij}\). For example, element at the 2\(^{nd}\) row and 2\(^{nd}\) column is \(a_{22}\).
Matrix Multiplication

The product of matrix \([A]\) of size \((m \times n)\) and matrix \([B]\) of size \((n \times p)\) will result in matrix \([C]\), with size \((m \times p)\).

\[
\begin{bmatrix}
A \\
(m \times n)
\end{bmatrix}
\begin{bmatrix}
B \\
(n \times p)
\end{bmatrix}
= \begin{bmatrix}
C \\
(m \times p)
\end{bmatrix}
\tag{2-4}
\]

**Note:** The \((ij)\)th component of \([C]\), i.e. \(c_{ij}\), is obtained by taking the DOT product,

\[
c_{ij} = (i\text{th row of } [A]) \cdot (j\text{th column of } [B])
\tag{2-5}
\]

**Example:**

\[
\begin{bmatrix}
2 & 1 & 3 \\
0 & -2 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 4 \\
5 & -2 \\
0 & 3
\end{bmatrix}
= \begin{bmatrix}
7 & 15 \\
-10 & 7
\end{bmatrix}
\]

\((2 \times 3)\) \(\times\) \((3 \times 2)\) \(=\) \((2 \times 2)\)
Matrix Transposition

If matrix \([A] = [a_{ij}]\), then transpose of \([A]\), denoted by \([A]^T\), is given by \([A]^T = [a_{ji}]\). Thus, the rows of \([A]\) becomes the columns of \([A]^T\).

**Example:**

\[
[A] = \begin{bmatrix}
1 & -5 \\
0 & 6 \\
-2 & 3 \\
4 & 2 \\
\end{bmatrix}
\]

Then,

\[
[A]^T = \begin{bmatrix}
1 & 0 & -2 & 4 \\
-5 & 6 & 3 & 2 \\
\end{bmatrix}
\]

**Note:** In general, if \([A]\) is of dimension \((m \times n)\), then \([A]^T\) has the dimension of \((n \times m)\).
Transpose of a Product

The transpose of a product of matrices is given by the product of the transposes of each matrices, in reverse order, i.e.

\[
\]  
(2-6)

Determinant of a Matrix

Consider a 2 x 2 square matrix \([x]\),

\[
[x] = \begin{bmatrix}
x_{11} & x_{12} \\
x_{21} & x_{22}
\end{bmatrix}
\]

The determinant of this matrix is give by,

\[
det[x] = x_{11}x_{22} - x_{21}x_{12}
\]  
(2-7)
EXAMPLE

Given that:

\[
[A] = \begin{bmatrix} 1 & 0 \\ -1 & 4 \end{bmatrix} \quad [C] = \begin{bmatrix} 3 & 1 & 0 \\ -1 & 0 & 3 \end{bmatrix}
\]

\[
[D] = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 4 & 0 \\ 2 & 0 & 3 \end{bmatrix} \quad \{E\} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}
\]

Find the product for the following cases:

a) \([A][C]\)

b) \([D]\{E\}\)

c) \([C]^T[A]\)
Solution of System of Linear Equations

System of linear algebraic equations can be solved for the unknown using the following methods:

a) Cramer’s Rule
b) Inversion of Coefficient Matrix
c) Gaussian Elimination
d) Gauss-Seidel Iteration

Example: Solve the following SLEs using Gauss elimination method.

(i) \[2x_1 + 1x_2 - 3x_3 = 11\]
(ii) \[4x_1 - 2x_2 + 3x_3 = 8\]
(iii) \[-2x_1 + 2x_2 - 1x_3 = -6\]
Gauss Elimination Method

Reducing a set of $n$ equations in $n$ unknowns to an equivalent triangular form (forward elimination). The solution is determined by back substitution process.

Basic approach

- Any equation can be multiplied (or divided) by a nonzero scalar
- Any equation can be added to (or subtracted from) another equation
- The position of any two equations in the set can be interchanged
Example: Solve the following SLEs using Gaussian elimination.

\[
\begin{align*}
2x_1 + 1x_2 - 3x_3 &= 11 \quad (i) \\
4x_1 - 2x_2 + 3x_3 &= 8 \quad (ii) \\
-2x_1 + 2x_2 - 1x_3 &= -6 \quad (iii)
\end{align*}
\]

Eliminate \(x_1\) from eq.(ii) and eq.(iii). Multiply eq.(ii) by 0.5 we get,

\[
\begin{align*}
2x_1 + 1x_2 - 3x_3 &= 11 \quad (i) \\
2x_1 - 1x_2 + 1.5x_3 &= 4 \quad (ii)^* \\
-2x_1 + 2x_2 - 1x_3 &= -6 \quad (iii)
\end{align*}
\]
Subtract eq. (ii)* from eq. (i), we obtain

\[ 2x_1 + 1x_2 - 3x_3 = 11 \]  
\[ 0x_1 + 2x_2 - 4.5x_3 = 7 \]  
\[ -2x_1 + 2x_2 - 1x_3 = -6 \]

Add eq. (iii) with eq. (i), yields

\[ 2x_1 + 1x_2 - 3x_3 = 11 \]  
\[ 0x_1 + 2x_2 - 4.5x_3 = 7 \]  
\[ 0x_1 + 3x_2 - 4x_3 = 5 \]
Eliminate $x_2$ from eq. (iii)*. Multiply eq. (ii)** by 3 and eq. (iii)* by 2 we get

\[2x_1 + 1x_2 - 3x_3 = 11 \quad (i)\]
\[0x_1 + 6x_2 - 13.5x_3 = 21 \quad (ii)**\]
\[0x_1 + 6x_2 - 8x_3 = 10 \quad (iii)**\]

Subtract eq. (iii)** from eq. (ii)***, we obtain

\[2x_1 + 1x_2 - 3x_3 = 11 \quad (i)\]
\[0x_1 + 2x_2 - 4.5x_3 = 7 \quad (ii)**\]
\[0x_1 + 0x_2 - 5.5x_3 = 11 \quad (iii)***\]

From eq. (iii)*** we determine the value of $x_3$, i.e.

\[x_3 = \frac{11}{-5.5} = -2\]
Back substitute value of $x_3$ into eq.(ii)** and solve for $x_2$, we get

$$x_2 = \frac{7 + 4.5(-2)}{2} = -1$$

Back substitute value of $x_2$ and $x_3$ into eq.(i) and solve for $x_1$, we get

$$x_1 = 6$$
Example

Solve the following systems of linear equations by using the Gaussian elimination method.

a) \[-x_1 + 3x_2 - 2x_3 = 2\]
   \[2x_1 - 4x_2 + 2x_3 = 1\]
   \[0x_1 + 4x_2 + x_3 = 3\]

b) \[2x_1 + x_2 - 3x_3 = 11\]
   \[4x_1 - 2x_2 + 3x_3 = 8\]
   \[-2x_1 + 2x_2 - 2x_3 = -6\]
Steps in solving a continuum problem by FEM

- Identify and understand the problem
  *(This essential step is not FEM)*

- Select the solution domain
  *Select the solution region for analysis.*

- Discretize the continuum
  *Divide the solution region into finite number of elements, connected to each other at specified points / nodes.*

- Select interpolation functions
  *Choose the type of interpolation function to represent the variation of the field variables over the element.*

- Derive element characteristic matrices and vectors
  *Employ direct, variational, weighted residual or energy balance approach.*

\[ [k]^{(e)} \{\phi\}^{(e)} = \{f\}^{(e)} \]
Steps … (Continued)

- Assemble the element characteristic matrices and vectors
  - *Combine the element matrix equations and form the matrix equations expressing the behavior of the entire solution region / system.*
  - *Modify the system equations to account for the boundary conditions of the problem.*

- Solve the system equations
  - *Solve the set of simultaneous equations to obtain the unknown nodal values of the field variables.*

- Make additional computations, if desired
  - *Use the resulting nodal values to calculate other important parameters.*
What is the problem?

Stress concentration along a shaft

Other examples:
-Scratches on a tensile surface
-Oil groove on a shaft
-Threaded connections
-Rivet holes under tension

Task:
To simulate stress and strain fields in the vicinity of a sharp notch under tensile load
### INTRODUCTION AND OVERVIEW

#### MMJ1153 – COMPUTATIONAL METHOD IN SOLID MECHANICS

**FE procedures:**

- Select the solution domain
- Discretize the continuum
- Choose interpolation functions
- Derive element characteristic matrices and vectors
- Assemble element characteristic matrices and vectors
- Solve the system equations
- Make additional computations, if desired

**FE software user steps:**

- **Draw the model geometry**
- **Mesh the model geometry**
- **Select element type**
- **(The FEA software was written to do this)**
  - Input material properties
- **(The computer will assemble it)**
  - Input specified load and boundary conditions
- **(The compiler will solve it)**
  - Request for output
- **Post-process the result file**
EXAMPLE: Stresses in a C(T) specimen

- Draw the model geometry
- Mesh the model geometry
- Select element type
  [20-node Hexahedral elements]
- Input material properties
  [Elastic modulus, Poisson’s ratio]
- Input specified load and BC
  [Pin loading, displacement rate, zero displacement at lower pin]
- Request for output
  [Displacement, strain and stress components]
- Post-process the result file
EXAMPLE: Stresses in a C(T) specimen

- Check for obvious mistakes/errors
- Extract useful information
- Explain the physics/mechanics
- Validate the results
MODELING CAPABILITIES – *Microelectronic device reliability*

- Prediction of spatial distribution of physical parameters
- Prediction of damage evolution characteristics

INTRODUCTION AND OVERVIEW

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INTRODUCTION AND OVERVIEW

- 14 layers [0/45/90/-45/45/-45/0°]s
- Layer thickness = 0.3571 mm

Comparison of deflection

<table>
<thead>
<tr>
<th>Material</th>
<th>Comp-0°</th>
<th>Comp-90°</th>
</tr>
</thead>
<tbody>
<tr>
<td>E</td>
<td>$E_{11}$</td>
<td>$E_{22}$</td>
</tr>
<tr>
<td>70 GPa</td>
<td>44.74 GPa</td>
<td>12.46 GPa</td>
</tr>
</tbody>
</table>