These slides are designed based on the book:
Course Content:

A – INTRODUCTION AND OVERVIEW
Numerical method and Computer-Aided Engineering; Physical problems; Mathematical models; Finite element method.

B – REVIEW OF 1-D FORMULATIONS
Elements and nodes, natural coordinates, interpolation function, bar elements, constitutive equations, stiffness matrix, boundary conditions, applied loads, theory of minimum potential energy; Plane truss elements; Examples.

C – PLANE ELASTICITY PROBLEM FORMULATIONS
Constant-strain triangular (CST) elements; Plane stress, plane strain; Axisymmetric elements; Stress calculations; Programming structure; Numerical examples.
D – ELASTIC-PLASTIC PROBLEM FORMULATIONS
Iterative solution methods, 1-D problems, Mathematical theory of plasticity, matrix formulation, yield criteria, Equations for plane stress and plane strain case; Numerical examples

E – PHYSICAL INTERPRETATION OF FE RESULTS
Case studies in solid mechanics; FE simulations in 3-D; Physical interpretation; FE model validation.
OBJECTIVE

To learn the use of finite element method for the solution of problems involving elastic-plastic materials.
Stress-strain diagram

• Effects of high straining rates
• Effects of test temperature
• Foil versus bulk specimens
Elastic-Plastic Behavior

Characteristics:

1. An initial elastic material response onto which a plastic deformation is superimposed after a certain level of stress has been reached.

2. Onset of yield governed by a yield criterion.

3. Irreversible on unloading.

4. Post-yield deformation occurs at reduced material stiffness.
Monotonic stress-strain behavior

Typical low carbon steel

Engineering stress \[ S = \frac{P}{A_o} \]

True stress \[ \sigma = \frac{P}{A} \]

Engineering strain \[ e = \frac{(l - l_o)}{l_o} = \frac{\Delta l}{l_o} \]

True or natural strain

\[ d\varepsilon = \frac{dl}{l} \quad \varepsilon = \int \frac{dl}{l} = \ln \left( \frac{l}{l_o} \right) \]

For the assumed constant volume condition, \[ A_l = A_o \cdot l_o \]

Then;

\[ \sigma = S(1 + e) \]

\[ \varepsilon = \ln \frac{A_o}{A} = \ln(1 + e) \]
Plastic strains

Elastic region – Hooke’s law:

\[ \sigma = E \varepsilon \]

\[ \log \sigma = \log E + (1) \log \varepsilon \]

Plastic region:

\[ \sigma = K (\varepsilon_p)^n \]

\[ \log \sigma = \log K + n \log \varepsilon \]

Engineering stress-strain diagram

True stress-strain diagram
Plastic strains - Example

SS316 steel

Non-linear /Power-law

\[ \sigma = K(\varepsilon_p)^n \]

\[ \sigma = 747.3\varepsilon_p^{0.199} \]
Ideal Materials behavior

- Elastic-perfectly plastic
- Perfectly plastic

- Elastic-linear hardening
Simulation of elastic-plastic problem

Requirements for formulating the theory to model elastic-plastic material deformation:

- An explicit stress-strain relationship under elastic condition.
- A yield criterion indicating the onset of plastic flow.
- Stress-strain relationship for post yield behavior.
Elastic-plastic problem in 2-D

In elastic region:

\[ \sigma_{ij} = C_{ijkl} \varepsilon_{kl} \]

\[ C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu \delta_{ik} \delta_{jl} + \mu \delta_{il} \delta_{jk} \]

\( \lambda, \mu \) are Lamé constants

Kronecker’s delta:

\[ \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \]
Elastic case for plane stress and plane strain (Review)

Strain

\[
\{\varepsilon\} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{bmatrix}
\]

Hooke’s Law

\[
\{\sigma\} = [D]\{\varepsilon\}
\]

Elasticity matrix

Plane stress

\[
[D] = \frac{E}{1-v^2} \cdot \begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix}
\]

Plane strain

\[
[D] = \frac{E}{(1+v)(1-2v)} \cdot \begin{bmatrix} 1-v & v & 0 \\ v & 1-v & 0 \\ 0 & 0 & \frac{1}{2} - v \end{bmatrix}
\]

Stress

\[
\{\sigma\} = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}
\]
Yield criteria:
Stress level at which plastic deformation begins

\[ f(\sigma_{ij}) = k(\kappa) \]

\( k(\kappa) \) is an experimentally determined function of hardening parameter

The material of a component subjected to complex loading will start to yield when the (parametric stress) reaches the (characteristic stress) in an identical material during a tensile test.

Other parameters:
Strain
Energy
Specific stress component (shear stress, maximum principal stress)
Yield criteria:
Stress level at which plastic deformation begins

\[ f(\sigma_{ij}) = k(\kappa) \]

Yield criterion should be independent of coordinate axes orientation, thus can be expressed in terms of stress invariants:

\[ J_1 = \sigma_{ii} \]
\[ J_2 = \frac{1}{2} \sigma_{ij} \sigma_{ij} \]
\[ J_3 = \frac{1}{3} \sigma_{ij} \sigma_{jk} \sigma_{ki} \]
Plastic deformation is independent of hydrostatic pressure

\[ f(J'_2,J'_3) = k(\kappa) \]

\( J'_2, J'_3 \) are the second and third invariant of deviatoric stress:

\[ \sigma'_{ij} = \sigma_{ij} - \frac{1}{3} \delta_{ij} \sigma_{kk} \]

\( \sigma_{ij} \) is the deviatoric stress
\( \sigma_{kk} \) is the hydrostatic stress
Yield Criteria

For ductile materials:
- Maximum-distortion (shear) strain energy criterion (von-Mises)
- Maximum-shear-stress criterion (Tresca)

For brittle materials:
- Maximum-principal-stress criterion (Rankine)
- Mohr fracture criterion
Maximum-distortion-energy theory (von Mises)

\[
(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2\sigma_Y^2
\]

For plane problems:

\[
\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 = \sigma_Y^2
\]
Maximum-normal-stress theory
(Rankine)

\[ |\sigma_1| = \sigma_{ult} \]
\[ |\sigma_2| = \sigma_{ult} \]
Tresca and von Mises yield criteria
Tresca and von Mises yield criteria
Strain Hardening behavior

The dependence of stress level for further plastic deformation, after initial yielding on the current degree of plastic straining

Strain Hardening models

Perfect plasticity
- Yield stress level does not depend on current degree of plastic deformation
Isotropic strain hardening
-Current yield surfaces are a uniform expansion of the original yield curve, without translation

-For strain soften material, the yield stress level at a point decreases with increasing plastic strain, thus the original yield curve contracts progressively without translation
-Consequently, the yield surface becomes a failure criterion
Strain Hardening models

Kinematic strain hardening
- Subsequent yield surface preserves their shape and orientation but translates in the stress space.
- This results in Bauschinger effect on cyclic loading
Bauschinger effect
Progressive development of yield surface

To relate the yield stress, $k$, to the plastic deformation by means of the hardening parameter, $\kappa$

Model 1: The degree of work hardening, $\kappa$, is expressed as a function of the total plastic work, $W_p$

$$
\kappa = W_p = \int \sigma_{ij} \left( d\varepsilon_{ij} \right)_p
$$

Increment of plastic strain

Model 2: The strain hardening, $\kappa$, is related to a measure of the total plastic deformation termed the effective, generalized or equivalent plastic strain:

$$
\kappa = \overline{\varepsilon}_p = \int d\overline{\varepsilon}_p
$$

Since yielding is independent of hydrostatic stress, $(d\varepsilon_{ii})_p = 0$, thus:

$$
(d\varepsilon'_{ij})_p = (d\varepsilon_{ij})_p
$$

$$
d\overline{\varepsilon}_p = \sqrt{\frac{2}{3}} \left( d\varepsilon_{ij} \right)_p \left( d\varepsilon_{ij} \right)_p
$$
Isotropic Hardening

For a stress state where $f = k$ represents the plastic state while elastic behavior is characterized by $f < k$, an incremental change in the yield function is:

$$df = \frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij}$$

$df < 0$  Elastic unloading

$df = 0$  Neutral loading (plastic behavior for perfectly-plastic material)

$df > 0$  Plastic loading (for strain hardening materials)
Elastic-plastic stress-strain relations

In plastic region:

\[ d\varepsilon_{ij} = (d\varepsilon_{ij})_e + (d\varepsilon_{ij})_p \]

Elastic strain increment:

\[ (d\varepsilon_{ij})_e = \frac{d\sigma'_{ij}}{2\mu} + \left( \frac{1 - 2\nu}{E} \right) \delta_{ij} d\sigma_{kk} \]

Plastic Flow Rule:
Assumes that the plastic strain increment is proportional to the stress gradient of the plastic potential, \( Q \):

\[ (d\varepsilon_{ij})_p \propto \frac{\partial}{\partial \sigma_{ij}} Q \]

\[ (d\varepsilon_{ij})_p = d\lambda \frac{\partial Q}{\partial \sigma_{ij}} \]

Plastic multiplier
Associated theory of plasticity

Since the *Flow Rule* governs the plastic flow after yielding, thus $Q$ must be a function of $J'_2$ and $J'_3$, similar to $f$, thus:

\[
\begin{align*}
    f & \equiv Q \\
    (d\varepsilon_{ij})_p &= d\lambda \frac{\partial f}{\partial \sigma_{ij}}
\end{align*}
\]

*Normality condition*
Associated theory of plasticity

Since the Flow Rule governs the plastic flow after yielding, thus $Q$ must be a function of $J_2'$ and $J_3'$, similar to $f$, thus:

$$f = Q$$

$$(d\varepsilon_{ij})_p = d\lambda \frac{\partial f}{\partial \sigma_{ij}}$$

Normality condition

Particular case of $f = J_2'$

$$\frac{\partial f}{\partial \sigma_{ij}} = \frac{\partial J_2'}{\partial \sigma_{ij}} = \sigma'_{ij}$$

$$(d\varepsilon_{ij})_p = d\lambda \sigma'_{ij}$$

Prandtl-Reuss equations
The incremental stress-strain relationship for elastic-plastic deformation:

\[ d\varepsilon_{ij} = (d\varepsilon_{ij})_e + (d\varepsilon_{ij})_p \]

\[ (d\varepsilon_{ij})_e = \frac{d\sigma'_{ij}}{2\mu} + \frac{(1-2\nu)}{E} \delta_{ij} d\sigma_{kk} + d\lambda \frac{\partial f}{\partial \sigma_{ij}} \]
Uniaxial elastic-plastic strain hardening behavior

Definition of term:

\[ E_T = \frac{d\sigma}{d\varepsilon} \]
Uniaxial elastic-plastic strain hardening behavior

Yield condition \( f(\sigma_{ij}) = k(\kappa) \)

Strain hardening hypothesis \( (\kappa) = \bar{\varepsilon}_p \)

\[
\overline{\sigma} = H(\bar{\varepsilon}_p) \\
\text{Proportional to } J'_2
\]

\[
\frac{d\overline{\sigma}}{d\bar{\varepsilon}_p} = H'(\bar{\varepsilon}_p)
\]
Uniaxial elastic-plastic strain hardening behavior

For uniaxial case \( \sigma_1 = \sigma \), \( \sigma_2 = \sigma_3 = 0 \)

\[
\overline{\sigma} = \sqrt{\frac{3}{2}} \sigma'_{ij} \sigma'_{ij} = \sigma
\]

\[
\left( d\varepsilon_1 \right)_p = d\varepsilon_p
\]

\[
\left( d\varepsilon_2 \right)_p = \left( d\varepsilon_3 \right)_p = 0.5d\varepsilon_p
\]

\[
d\overline{\varepsilon}_p = \sqrt{\frac{2}{3}} \left( \varepsilon'_{ij} \right)_p \left( \varepsilon'_{ij} \right)_p = d\varepsilon_p
\]

Plastic straining is assumed to be incompressible, \( \nu = 0.5 \)

- The hardening function, \( H \) is determined experimentally from tension test data.
- \( H' \) is required
Uniaxial stress-strain curve for elastic-linear hardening

\[ E_T = \frac{d\sigma}{d\varepsilon} \]

Assume \[ d\varepsilon = d\varepsilon_e + d\varepsilon_p \]

Strain-hardening parameter:

\[ H' = \frac{d\sigma}{d\varepsilon_p} \]
Numerical solution process for non-linear problems
Notations:

Linear elastic problem:

\[ KQ + F = 0 \]

\[ [K]\{Q\} + \{F\} = \{0\} \]

Nonlinear problem:

\[ H\phi + f = 0 \]

\[ [H(\phi)]\{\phi\} + \{f\} = \{0\} \]

Thus, iterative solution is required

For 1-D (1 variable) problem:

\[ H(\phi)\phi + f = 0 \]
Iterative Solution Methods

The problem:

\[ H\varphi + f = 0 \]

\[ \begin{bmatrix} H(\varphi) \end{bmatrix} \{ \varphi \} + \{ f \} = \{ 0 \} \]

\[ H(\phi)\phi + f = 0 \quad \text{For single variable} \]

- Direct iteration / successive approximation
- Newton-Raphson method
- Tangential stiffness method
- Initial stiffness method
Iterative Solution Methods
(a) Direct iteration or successive approximation

In each solution step, the previous solution for the unknowns \( \varphi \) is used to predict the current values of the coefficient matrix \( H(\varphi) \)

\[
H(\varphi) \varphi + f = 0
\]

\[
\varphi = -[H(\varphi)]^{-1} f
\]

For \((r+1)^{th}\) approximation:

\[
\varphi^{r+1} = -[H(\varphi^r)]^{-1} f
\]

Initial guess \( \varphi^0 \) is based on solution for an average material properties.

Convergence is not guaranteed and cannot be predicted at initial solution stage. Converged iteration when \( \varphi^{r-1} \) and \( \varphi^r \) are close.
Task: to illustrate the process.

Direct iteration method for a single variable
Convex $H$-$\phi$ relation
Steps for Direct Iteration Method

Start with \( \{ \phi^0 \} \)

Determine \([H(\phi^0)]\)

Calculate \( \{ \phi^1 \} = -[H(\phi^0)]^{-1} \{ f \} \)

Repeat until \( \{ \phi^{r+1} \} \approx \{ \phi^r \} \)
Direct iteration method for a single variable
Concave $H$-$\phi$ relation
Assignment # 5

A one degree of freedom problem can be represented by

\[ H\phi + f = 0 \]

where \( f = 10 \) and \( H(\phi) = 10 (1+e^{3\phi}) \)

Design and run a computer algorithm to determine the solution of the problem using direct iteration method and Newton-Raphson method.

(a) Show the flow chart of the algorithm and description of the steps.
(b) Tabulate the iterative values of \( H \) and \( \phi \) and compare the converged solution from the two methods. Specify a convergence tolerance of 1%.

Try with \( \phi^0 = 0.2 \)
Should converge at \( \phi^7 = -9.444 \)
Review of Newton-Raphson Method
(for finding root of an equation $f(x) = 0$)

Taylor Series expansion:

$$f(x) = f(x_1) + (x - x_1)f'(x_1) + \frac{1}{2!}(x - x_1)^2 f''(x_1) + ...$$

Consider only the first 2 terms:

$$f(x) \approx f(x_1) + (x - x_1)f'(x_1)$$

Set $f(x) = 0$ to find roots:

$$f(x_1) + (x - x_1)f'(x_1) = 0$$

$$x = x_1 - \frac{f(x_1)}{f'(x_1)}$$
Review of Newton-Raphson Method
(for finding root of an equation \( f(x) = 0 \))

Iterative procedures:

\[
x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}
\]
Newton-Raphson Method

The problem

\[ \Psi = H\varphi + f \neq 0 \]
\[ \{\psi(\varphi)\} = [H(\varphi)]\{\varphi\} + \{f\} \neq \{0\} \]

If true solution exists at \( \varphi^r + \Delta \varphi^r \)

\[ \psi^r_i = -\sum_{j=1}^{N} \Delta \phi^r_j \left( \frac{\partial \psi^r_i}{\partial \phi_j} \right)^r \]
\[ \{\psi(\varphi^r)\} = -[J(\varphi^r)]\{\Delta \varphi^r\} \]

Initial guess is based on \( \varphi^0 \) is based on solution for an average material properties.

Converged iteration when \( \varphi^{n-1} \) and \( \varphi^n \) are close
Jacobian Matrix

\[ J_{ij} = \left( \frac{\partial \psi_i}{\partial \phi_j} \right) = h_{ij}^r + \sum_{k=1}^{m} \left( \frac{\partial \psi_i}{\partial \phi_j} \right)^r \phi_k^r \]

\[ [J(\varphi)] = [H(\varphi)] + [H'(\varphi)] \]

\[ \{\Delta \varphi^r\} = -[J(\varphi^r)]^{-1} \{\psi(\varphi^r)\} \]
\[ = -\left[H(\varphi^r) + [H'(\varphi^r)]^{-1} \{\psi(\varphi^r)\} \right] \]

\[ \{\varphi^{r+1}\} = \{\varphi^r\} + \{\Delta \varphi^r\} \]
Newton-Raphson method for a single variable
Newton-Raphson method for a single variable
Tangential Stiffness Method 
(*Generalized Newton-Raphson Method*)

Linearize \{\varphi\} in \[H(\varphi)\] such that the term \[H'(\varphi)\] can be omitted

Steps:

\[ [H(\varphi)]\{\varphi\} + \{f\} = \{\psi(\varphi)\} \]

Assume a trial value \{\varphi^0\}

Calculate \([H(\varphi^0)]\]

Calculate \{\psi(\varphi^0)\}

Calculate \[\{\Delta \varphi^0\} = -[H(\varphi^0)]^{-1}\{\psi(\varphi^0)\} \]

\[\{\varphi^1\} = \{\varphi^0\} + \{\Delta \varphi^0\}\]

Iterate until \{\psi^r\} \approx \{0\}
Tangential stiffness method for a single variable
Initial Stiffness Method

Recall that in Direct Stiffness Method: \[ \{ \varphi^{r+1} \} = -[H(\varphi^r)]^{-1} \{ f \} \]

In Tangential Stiffness Method: \[ \{ \Delta \varphi^r \} = -[H(\varphi^r)]^{-1} \{ \psi(\varphi)^r \} \]

This requires complete reduction and solution of the set of simultaneous equations for each iteration.

Use the initial stiffness \([H(\varphi^0)]\) for subsequent approximation.

\[ \{ \Delta \varphi^r \} = -[H(\varphi^0)]^{-1} \{ \psi(\varphi)^r \} \]
Initial stiffness method for a single variable
Uniaxial stress-strain curve for elastic-linear hardening

Assume \( d\varepsilon = d\varepsilon_e + d\varepsilon_p \)

Strain-hardening parameter:

\[
H' = \frac{d\sigma}{d\varepsilon_p}
\]
Element stiffness for elastic-plastic material behavior

![Element diagram](image)

Elastic behavior:

\[ K_e = \frac{F}{\delta} = \frac{EA}{L} \]

\[ [K_e]^e = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \]

Elastic-plastic behavior:

\[ \varepsilon = \frac{\delta}{L} \ ; \ d\delta = d\varepsilon L = (d\varepsilon_e + d\varepsilon_p) L \]

\[ \sigma = \frac{F}{A} \ ; \ dF = d\sigma \cdot A = H' d\varepsilon_p A \]

\[ K_{ep} = \frac{dF}{d\delta} = \frac{H' d\varepsilon_p A}{\left( \frac{d\sigma}{E} + d\varepsilon_p \right) L} = \frac{EA}{L} \left( 1 - \frac{E}{E + H'} \right) \]

\[ [K_{ep}]^e = \frac{EA}{L} \left( 1 - \frac{E}{E + H'} \right) \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \]

Element stiffness matrix for elastic-plastic material behavior
Numerical singularity issue

After initial yielding, $H'=0$, 
Thus, $[K_{ep}]^{(e)} = [0]$ 

Use initial stiffness method to ensure positive definite $[K]$
Program structure for 1-D problem

**Input geometry, loading, b.c.**

**INITIALIZATION**
Zero all arrays for data storing

**INC. LOAD**

**SET INDICATOR**
Choose type of solution algorithm – direct iteration, tangential stiffness, etc

**STIFFNESS**

\[ [K]^{(e)} \]

\[ [K] \{\varphi\} = \{F\} \]

**ASSEMBLY**

**REDUCTION**
Solve for \( \{\varphi\} \)

**RESIDUAL**
Calculate residual force \( \{\psi\} \) for Newton-Raphson, Initial and Tangential stiffness method

**Check for convergence**

**Output results**
Incremental stress and strain changes at initial yielding

Task: to illustrate the process.

\[
\psi^{r-1} = H \phi^{r-1} + f
\]

\[
\Delta \phi^r = - \left[ H(\phi)^r \right]^{-1} \psi(\phi^r)
\]

\[
\Delta \varepsilon^r = E \Delta \varepsilon^r
\]

\[
\sigma_e^r = \sigma_e^{r-1} + \Delta \sigma_e^r
\]

Check if the element has previously yielded:

\[
\sigma_y^{r-1} = \sigma_y + H' \varepsilon_p^{r-1}
\]
Engineering stress-strain curve for Al2219