Standard Form of Eigenvalue Problem

\[
[A] \{\tilde{X}\} = \lambda \{\tilde{X}\}
\]

General Form of Eigenvalue Problem

\[
[A] \{\tilde{X}\} = \lambda [B] \{\tilde{X}\}
\]

General form can be converted to standard form (solution methods employ standard form)
Conversion to Standard Form

• When \([B]\) is diagonal
Conversion to Standard Form

- When $[B]$ is symmetric & positive definite
Methods for Eigenvalue Problems

- Transformation
  - Jacobi
  - Givens
  - Householder

- Iterative
  - Power method
  - Vector iteration, subspace iteration, etc.

- Determinant search method
  - Characteristic polynomial
Power Method

• Can be used to get largest eigenvalue
• Lowest & other eigenvalues obtained with slight modifications
• Eigenvectors obtained as by-product (no separate steps needed)
Inverse Power Method

- Get the smallest eigenvalue
- Apply power method on \([A]^{-1}\)
- Iteration will converge to largest value of \(1/\lambda\) (or smallest \(\lambda\))
Intermediate Eigenvalues

- **Deflation** – removing the highest known eigenvalue (so that process converges on next largest eigenvalue)
- Get new matrix $[A]_{\text{new}}$ with largest eigenvalue removed
- Hotelling's method (for symmetric matrices)
- Based on orthogonality of symmetric matrices

\[
\{X\}_{i}^{T}\{X\}_{j} = \begin{cases} 
0 & \text{for } i \neq j \\
1 & \text{for } i = j
\end{cases}
\]
• Normalizing eigenvector \(\{X\}\) by dividing each element by normalizing factor
\[
\frac{\sqrt{\sum_{k=1}^{n} x_k^2}}{n}
\]
so that \(\{X\}^T\{X\} = 1\)

• So, new matrix \([A]_2 = [A]_1 - \lambda_1\{X\}_1^T\{X\}_1\)

• Do power method on new matrix \([A]_2\) & get next largest \(\lambda_2\)

• Repeat for other eigenvalues
Determinant search

- For non-trivial solutions of \([A] - \lambda[I] \{\hat{X}\} = \{\hat{0}\}\) \[\det([A] - \lambda[I]) = 0\]

- The expression of determinant will result in a polynomial of order n in \(\lambda\) (characteristic polynomial)

\[(-1)^n(\lambda^n - p_1\lambda^{n-2} - \cdots - p_{n-1}\lambda - p_n) = 0\]

- Roots of this polynomial are the eigenvalues
Faddeev-Leverrier Method

- For large order $[A]'s$, expanding determinant to get the polynomial is tedious
- Use Faddeev-Leverrier Method to get polynomial
Faddeev-Leverrier Method

- Get polynomial coefficients $p_1, p_2, \ldots, p_n$ from sequence of matrices $[P_i]$
  
  $[P_1] = [A], \ p_1 = \text{trace}[P_1]$
  
  $[P_2] = [A]([P_1] - p_1[I]), \ p_2 = \frac{1}{2}\text{trace}[P_2]$
  
  $[P_3] = [A]([P_2] - p_2[I]), \ p_3 = \frac{1}{3}\text{trace}[P_3]$

- succinctly;
  
  $[P_i] = [A]([P_{i-1}] - p_{i-1}[I]), \ p_i = \frac{1}{i}\text{trace}[P_i]$
  
  $\vdots$
  
  $[P_n] = [A]([P_{n-1}] - p_{n-1}[I]), \ p_n = \frac{1}{n}\text{trace}[P_n]