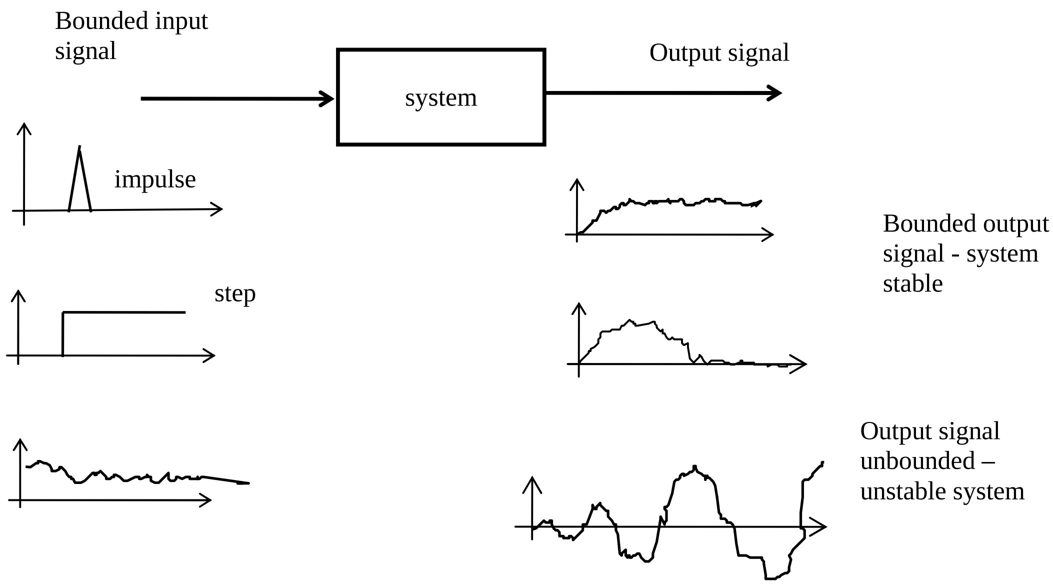
Stability of Linear Control System

Concept of Stability

Closed-loop feedback system is either stable or unstable. This type of characterization is referred to as *absolute stability*. Given that the system is stable, the degree of stability of the system is referred to as *relative stability*.

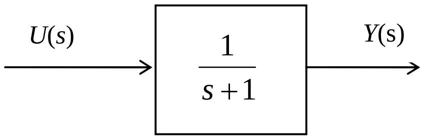
A *stable system* is defined as a system with bounded response to a bounded input.



Consider the concept of stability for cones shown below.

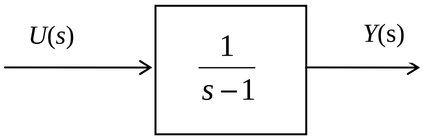
Example 3.1

Obtain the time response of the system shown below for a unit step input $U(s) = 1/s$ (an example of a bounded input). Determine whether the system is stable or unstable.



Example 3.2

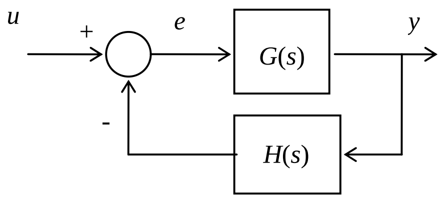
Obtain the time response of the system shown below for a unit step input $U(s) = 1/s$ (an example of a bounded input). Determine whether the system is stable or unstable.



Stability of a Closed-loop system

When a closed-loop is designed, the problem of stability may arise if the controller is not properly designed. A stable open-loop system may become an unstable closed-loop system. In some cases, an unstable open-loop system can be stabilized using a feed-back control system.

Consider a closed-loop system shown below.



The transfer function is

$$\frac{y}{u} = \frac{G}{1 + GH}$$

The equation 1 + *GH* = 0 is known as the *characteristic equation*. In general, the transfer function of a closed loop system can be written as

$$\frac{y}{u} = \frac{N(s)}{D(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_0}, \quad n > m$$

where $D(s) = 0$ is the characteristic equation. This transfer function can be written in *pole-zero configuration* as

$$\frac{y}{u} = \frac{(s + z_1)(s + z_2) \cdots (s + z_m)}{(s + p_1)(s + p_2) \cdots (s + p_n)}$$

where the *poles* are the roots of the denominator of the transfer function and *zeros* are the roots of the numerator of the transfer function.

The response of this system to a unit impulse input $U(s) = 1$ can be obtained as

The response is bounded if the poles are *negative*. Stability of the system is determined by the poles only. Thus, the sufficient condition for stability of a feedback control system is *all poles of the closed loop transfer function must have a negative real values.* Stability region on the *s*-plane is shown below. The system is stable if all the poles are located on the left hand side of the imaginary axis.

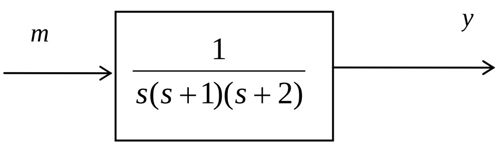
Example 3.3

Determine the stability of a system with a characteristic equation

$$q(s) = s^3 + 4s^2 + 6s + 4$$

Example 3.4

For a marginally stable open-loop system shown below, determine the stability of a closed-loop system if a proportional controller gain $K_p = 24$ is used.



Routh Stability Criterion

The *Routh Stability Criterion* is a method which can determine the existence of positive poles. This criterion is sufficient if the designer only wish to determine the range of control parameter that will ensure closed loop stability. Consider a closed loop system with the characteristic equation

$$q(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0 = 0$$

The stability of this system can be tested by constructing the Routh table as shown below.

This table is filled horizontally and vertically until the remaining elements are zeros. The characteristic equation has all negative roots if the signs of all elements in the first column are the same. The number of positive roots is equal to the number of the signs change.

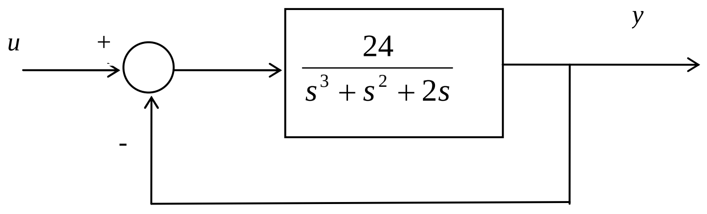
Example 3.5

Determine the stability of a system with a characteristic equation

$$q(s) = s^4 + 5s^3 + 20s^2 + 40s + 50$$

Example 3.6

Determine the stability of a closed-loop system shown below.

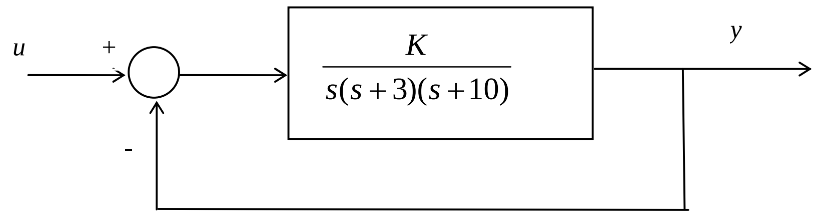


Example 3.7

Determine the stability of a system with a characteristic equation

$$q(s) = s^5 + 2s^4 + 2s^3 + 4s^2 + 11s + 10$$

Example 3.8



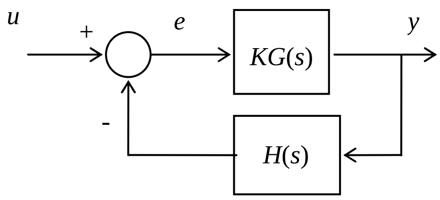
Determine the range of K which ensure the closed loop system to be stable.

The Root Locus Method

The Root Locus Concept

The relative stability and the transient performance of a closed-loop control system are directly related to the location of the closed-loop roots of the characteristic equation in the s-plane. It is frequently necessary to adjust one or more parameters in order to obtain suitable root locations. *The root locus is the path of the roots of the characteristic equation traced out in the s-plane as a system parameter is changed.*

Consider a feedback control system shown below.



The closed-loop transfer function, the open-loop transfer function, and the characteristic equation can be written as:

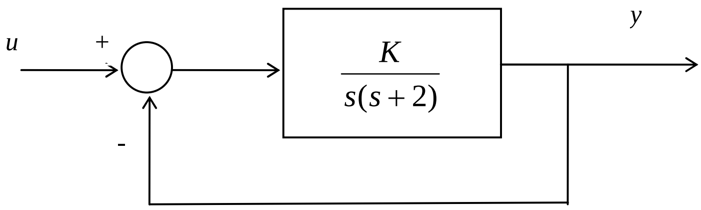
It can be seen that the values of the roots of the characteristic equation will change if the value of the parameter K is changed. When $K = 0$ the locus starts at the poles of the open-loop transfer function and the locus ends at the zeros of the open-loop transfer function when $K = \infty$.

Magnitude and Angle Criteria

Every point on the locus must satisfy the magnitude and angle criteria and can be formally written as:

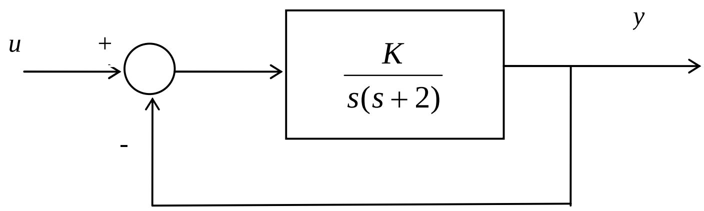
Example 3.9

Draw the locus of the roots of the characteristic equation of the control system shown below when K varies from 0 to ∞.



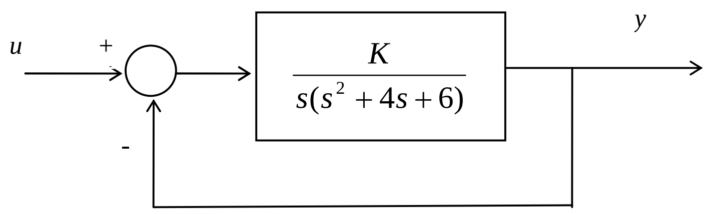
Example3.10

For a control system shown below, show that the root locus starts at the poles of the open-loop transfer function when K = 0 and it ends at the zeros of the open-loop transfer function when $K = \infty$.



Example 3.11

Using the magnitude and angle criteria, verify that s_1 = -1 + j is one of the roots of the characteristic equation of the control system shown below when K = 4



The Root Locus Procedure

The Root Locus Procedure

1. Locate the poles and zeros of the open-loop transfer function and plot them using x for poles and o for zeros.
2. The root locus on the real axis always lies in a section of the real axis to the left of an odd number of poles and zeros.
3. The loci begin at the poles and end at zeros or zeros at infinity, ∞, along asymptotes.
4. The number of asymptotes is equal to the number of poles minus the number of zeros, $p - z$. The direction of the asymptotes is define by the asymptote angle ϕ, where

   $$\phi = \frac{(2r+1)180^0}{p-z} \qquad \text{with} \quad r = 0, 1, 2, \ldots$$

5. All asymptotes intersect the real axis at a single point σ_A, often called the asymptote centroid, defined by

   $$\sigma_A = \frac{\sum p_i - \sum z_i}{p-z}$$

   where p_i is the values of the poles

   z_i is the values of the zeros

6. Points of breakaway from or arrival at the real axis may also exist and can be obtained by rearranging the characteristic equation to isolate the multiplying factor K in the form of

   $$K = f(s)$$

   On the real axis, the breakaway point happens when K is maximum, i.e.

   $$\frac{dK}{ds} = 0$$

7. The loci are symmetrical about the real axis. The loci approach or leaving the real axis at an angle of $\pm 90^o$.
8. The angle of departure, θ_d, form a complex pole is obtained from the angle criterion

$$\theta_d = 180^0 - \Sigma\theta_p + \Sigma\theta_z$$

   where θ_p and θ_z are the angles from other poles and zeros to the pole in question.
9. The intersection of the locus with the imaginary axis is obtained using the Routh stability criterion.

Example 3.12

Draw the root locus for a control system with an open-loop transfer function given as

$$KGH(s) = \frac{K}{s(s+2)(s+6)}$$

Determine the number of asymptotes, the asymptote angles, and the asymptote centroid.

Example 3.13

Draw the root locus for a control system with an open-loop transfer function given as

$$KGH(s) = \frac{K}{(s+2)(s+4)}$$

Determine the breakaway point.

Example 3.14

Draw the root locus for a control system with an open-loop transfer function given as

$$KGH(s) = \frac{K(s+2)}{(s^2+2s+4)}$$

Determine the breakaway point.

Example 3.15

Draw the root locus for a control system with an open-loop transfer function given as

$$KGH(s) = \frac{K(s+2)}{s(s^2+2s+2)}$$

Determine the departure angle from the complex poles.

Example 3.16

Draw the root locus for a control system with an open-loop transfer function given as

$$KGH(s) = \frac{K}{s(s^2 + 6s + 25)}$$

Determine the intersection of the locus with the imaginary axis.

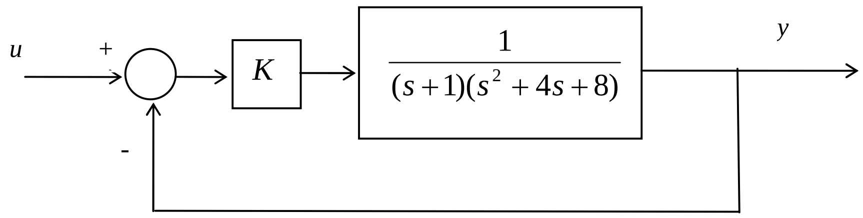
Interpretation of Root Locus

Interpretation of Root Locus and controller design

When the design of a feedback control system is undertaken, the controller must be chosen such that the closed-loop system is stable and its transient response is satisfactory and all specifications are satisfied. For a stable system, all roots must lie on the left-hand side of the imaginary axis. Meanwhile, the transient response of the closed-loop system is determined by the damping ratio, ξ , natural frequency, ω_n, damped natural frequency, ω_d , and time constant, T. The values of these parameters can be estimated from the location of the dominant roots from the root locus in the s-plane.

Example 3.17

A control system is represented by a block diagram shown below.



a) Draw the root locus when K increases from zero.

b) It is given that when K = 10, the roots of the characteristic equation are –3 dan –1 ± j2.24. Determined the damping ratio, natural frequency, damped natural frequency, and time constant, T for K = 10.

Example 3.18

For a control system shown below, chose the value of the controller K such that the maximum value of the damping ratio is 0.5 and the minimum value of the time constant is 1s.