

**Frequency Response Analysis**

**Frequency Response**
The performance of a control system is measured more realistically by its time-domain characteristics. The time response is usually more difficult to determine analytically especially for high-order system. A very practical and important alternative approach to the analysis and design is the *frequency response* method. To conduct a frequency-domain analysis does not imply that the system will be subject only to a sinusoidal input. Rather, from the frequency response studies, the time-domain performance can be projected. For example, the transfer function and relative stability of a system can be determine from the frequency response analysis.

The *frequency response* of a system is defined as the steady-state response of the system to a sinusoidal input signal. For a linear system, the resulting output signal is also sinusoidal; it differs from the input waveform only in amplitude and phase angle.

Consider a linear system shown below.

\[ u(t) = A \sin \omega t \quad \text{or} \quad U(s) = \frac{A \omega^2}{s^2 + \omega^2}. \]

It can be shown that the steady-state response is

\[ y_{ss} = B \sin (\omega t + \phi) \]
where $B$ and $\phi$ is obtained from the frequency response function $G(j\omega)$, that is by replacing $s = j\omega$ as shown below. The frequency response is obtained for all $\omega$ from 0 to $\infty$.

![Diagram showing input $u(t)$ and output $y(t)$ over time.]

**Example 4.1**

Obtain the steady state response for the system shown below if the input signal is $u(t) = 4 \sin 2t$.

\[ \frac{1}{s + 1} \]
Nyquist Diagram
The Nyquist diagram is the polar plot of the magnitude and phase angle of the open-loop transfer function $GH(j \omega)$ as $\omega$ varies from 0 to $\infty$.

Example 4.2
Draw the polar plot of the frequency response for a differentiator transfer function
$$G(s) = s$$

Example 4.3
Draw the polar plot of the frequency response for an integrator transfer function
$$G(s) = 1/s$$
Example 4.4
Draw the polar plot of the frequency response for a simple lag (first order) transfer function
\[ G(s) = \frac{1}{1 + Ts} \]

Example 4.5
Draw the polar plot of the frequency response for a simple lead (first order) transfer function
\[ G(s) = 1 + Ts \]
Example 4.6

Draw the Nyquist diagram for the second order open-loop transfer function

\[ GH(s) = \frac{\omega^2}{s^2 + 2\xi\omega_n s + \omega_n^2} \]
**Block diagram in series**

Consider a system represented by a block diagram shown below.

![Block Diagram](image_url)

The frequency response function, magnitude and phase angle can be obtained as follows:

Transfer function: \( G(s) = [G_1(s)] [G_2(s)] [G_3(s)] \)

System gain: \( |G(s)| = |G_1(s)||G_2(s)||G_3(s)| \)

Phase angle:

The transfer function of the system can also be written as a ratio of polynomials in \( s \) where the magnitude and phase angle can be obtained as follows:

A good Nyquist diagram is obtained when the frequency range is chosen properly. As a guideline, the frequency range is chosen such that
Example 4.7

Draw the polar plot of the frequency response for the system shown below.

\[ U(s) \rightarrow \frac{4}{s + 2} \rightarrow \frac{2}{s + 4} \rightarrow Y(s) \]
Example 4.8

Draw the Nyquist diagram for the control system shown below.
Nyquist Stability Criterion

Absolute and Relative Stability
The Routh stability criterion can only determine the absolute stability without any information about the relative stability. The Nyquist stability criterion not only can determine the absolute stability but also the relative stability of the control system.

The Nyquist stability criterion is obtained from the polar plot of the frequency response of the open-loop transfer function $GH(s)$ as shown below.

The closed-loop system is stable when $-1$ is located on the left-hand side of the polar plot of the open-loop transfer function when the frequency varies from $\omega = 0$ to $\omega = \infty$. The system is critically stable if the plot goes through $-1$. The system becomes unstable if $-1$ is located on the right-hand side of the plot.
Relative Stability

The relative stability is measured using the phase margin (jidar fasa), $\alpha$, and gain margin (jidar gandaan), $GM$. The system approaches the unstable state when $\alpha \to +0$ or $GM \to +0$. Consider the Nyquist diagram shown below.

The phase crossover frequency (frekuensi lintas fasa), $\omega_p$, is the frequency when $\angle GH(j\omega) = -180^\circ$. The gain crossover frequency (Frekuensi lintas gandaan), $\omega_g$, is defined as $|GH(j\omega_g)| = 1$.

As a guideline, the closed-loop system will have adequate damping if the gain margin and phase margin is between the interval shown below.
Example 4.9

For a control system shown below, obtain the phase margin and gain margin. Determine the phase crossover frequency and gain crossover frequency.
Example 4.10

Using the Nyquist stability criterion, determine the maximum value of the proportional controller $K$ for stability.
Bode Diagram

Bode diagram is an alternative to the Nyquist diagram. The Nyquist diagram is the polar plot of the frequency response function $GH(j\omega)$, while the Bode diagram is the plot of the gain $|GH(j\omega)|$ in decibels (dB) vs. the frequency $\omega$ and the plot of the phase angle $\phi(\omega)$ vs. the frequency $\omega$ on different set of axes on a semi-log graph as shown below. The logarithmic scale for frequency on the horizontal axes is the same for both plots (gain and phase angle).

The value in dB for the gain is defined as:

$$|G(j\omega)|_{\text{dB}} = 20\log|G(j\omega)|$$
If the transfer function $GH(s)$ is known, the gain and phase angle can be calculated identical to the calculations for drawing the Nyquist diagram. However, it is easier to draw the Bode diagram using the straight-line approximation.

For a transfer function with some combinations of simple leads and lags function, the Bode diagram is drawn by adding each lead and lag curves as shown below.
Example 4.11

Draw the Bode for the following transfer functions

a) \( G(s) = K \)  \quad b) \( G(s) = s \)  \quad c) \( G(s) = 1/s \)  \quad d) \( G(s) = 1/s^2 \)
Example 4.12

Draw the Bode diagram for the following transfer function:

a) a simple lag, $G(s) = 1/(1 + T_1 s)$  

b) a simple lead, $G(s) = 1 + T_2 s$
Example 4.13

Draw the Bode diagram for the second order open-loop transfer function

\[
GH(s) = \frac{\omega^2}{s^2 + 2\xi\omega_n s + \omega_n^2}
\]
Example 4.14

Draw the bode diagram for the transfer function

\[ G(s) = \frac{10}{s + 2} \]
Example 4.15

Draw the bode diagram for the transfer function

\[ G(s) = \frac{10}{s(s + 2)} \]
Example 4.16

Draw the bode diagram for the transfer function

\[ G(s) = \frac{10(1 + 0.5s)}{s(1 + 10s)(1 + 2s)} \]
Example 4.17

Draw the bode diagram for the transfer function

\[ G(s) = \frac{20}{s(1 + 0.1s + 0.25s^2)} \]
Estimation of the transfer function from Bode diagram

The Bode diagram of a system can be obtained experimentally using the frequency response method. From the Bode diagram, the transfer function of the system can be estimated as follows:

1. Approximate the logarithmic plot of the gain using the straight-line approximation.
2. The values of the frequency where the slope of the curve changes is the break frequency (a lag if the slope decreases by 20 dB/decade, a lead if the slope increases by 20 dB/decade).
3. If the slope decreases by 40 dB/decade, there exists either a second order lag or a double first order lags transfer function.
4. For a second order lag transfer function, the damping ratio is approximated from the gain magnitude at the natural frequency.

5. Check whether there exists an overall constant gain $K$ of the system.
Example 4.18

Approximate the transfer function of a system with the following frequency response data.

<table>
<thead>
<tr>
<th>ω (rad/s)</th>
<th>0.1</th>
<th>0.4</th>
<th>1.0</th>
<th>2.0</th>
<th>4.0</th>
<th>6.0</th>
<th>8.0</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>G(jω)</td>
<td></td>
<td>199</td>
<td>12</td>
<td>2.0</td>
<td>0.5</td>
<td>0.06</td>
</tr>
<tr>
<td>fasa (°)</td>
<td>-185</td>
<td>-198</td>
<td>-202</td>
<td>-225</td>
<td>-250</td>
<td>-258</td>
<td>-265</td>
</tr>
</tbody>
</table>
Stability Analysis Using Bode Diagram

The closed-loop system is stable if the logarithmic curve of the open-loop gain, $20\log \left| GH(j\omega) \right|$, crosses the 0 dB axis on the left-hand side of the phase crossover frequency. The Bode diagrams for stable, critically stable, and unstable closed-loop system are shown in the diagram below.

Relative stability

From the Bode diagram, the gain margin is the and the phase margin can be obtained as follows:
When a proportional controller is used, the limit of the proportionality constant $K$ for stability can be determined from the value of the gain margin. The gain curve can be shifted upward as much as the value of the gain margin before the closed-loop system reaches critical stability.
Example 4.19

For a system shown below, draw the Bode diagram for $K = 1$. From the Bode diagram, obtain the gain and phase margin. Estimate the critical value of $K$ for stability.

\[
\frac{20(s + 0.8)}{s^2(s + 2)(s + 8)}
\]