Abstract – Lattice Boltzmann numerical method was employed to simulate an incompressible fluid flow in enclosure. Thorough derivation of macroscopic hydrodynamics equations from the continuous Boltzmann equation is performed. After showing how the formulation of the mesoscale particle dynamics fits in to the framework of lattice Boltzmann simulations, numerical results of lid-driven flow inside square and triangular cavities are presented to highlight the applicability of the approach.

Keywords: Shear driven cavity, lattice Boltzmann method, distribution function, mesoscale method

1. Introduction

Computational Fluid Dynamics (CFD) has emerged as a powerful tool for the analysis of system involving fluid flow, heat transfer and associated phenomena such as chemical reactions, evaporation, condensation, etc [1-6]. From 1960s onwards, the aerospace industry has integrated CFD techniques into the design, research and development and manufacturing of aircraft and jet engines [7-9]. More recently, CFD has been applied to the design of internal combustion engines, combustion chambers of gas turbines and furnaces [10-12]. Furthermore, motor vehicle manufacturers now routinely predict drag forces, under-bonnet airflows and the in-car environment with CFD [13-15].

The fundamental law of any fluid flow problems is the Navier-Stokes equations, which define any single-phase fluid flow. These equations can be simplified by removing terms describing viscosity to yield the Euler equations. Further simplification, by removing terms describing vorticity yields the full potential equations. Finally, these equations can be linearized to yield the linearized potential equations.

Historically, finite difference method (FDM) [16] was the first computational method used by researchers to solve fluid flow and heat transfer problem by solving Navier-Stokes equation. However, due to the frustration on FDM, which cannot be effectively used on complex geometry, finite element method (FEM) has been introduced in 1950s [17]. In 1980s, finite volume method (FVM) was developed at Imperial College, mainly to solve fluid dynamic problems [18]. Since then the finite volume method is extensively used to solve transport phenomena problems. Indeed, the FDM, FEM and FVM belong to the same family of weighted residual method, however, they limit their simulation in the range of continuum fluid.

There are few numerical methods that simulate the evolution of fluid flow at particle level. Among them are Direct Simulation Monte Carlo (DSMC) [19] and Molecular Dynamic (MD) methods [20]. In these methods, the trajectories of every particle together with their position in the system are predicted using the second Newton’s law. From the knowledge of the forces on each atom, it is possible to determine the acceleration of each atom in the system. These methods are deterministic; once the positions and velocities of each atom are known, the state of the system can be predicted at any time in the future or the past. But remember, a cup of water contains $10^{23}$ number of molecules. Even when a gas is being considered where there are fewer molecules and a larger time-step can be used, because of the longer mean free path of the molecules, the number of molecules that can be considered is still limited. However, the question is do we really need to know the behavior of each molecule or atom? The answer is no. It is not important to know the behavior of each particle, it is important to know the function that can represent the behavior of many particles (mesoscale). As a result, in 1988, the lattice Boltzmann method (LBM) [21-23], a
mesoscale numerical method based on statistical distribution function has been introduced to replace MD and DSMC methods. Historically, LBM was derived from lattice gas automata (LGA) [24]. Consequently, LBM inherits some features from it precursor, the LGA method. The first LBM model was a floating-point version of its LGA counterpart. Each particle in LGA model (represented by single bit Boolean integer) was replaced by a single particle distribution function represented by a floating-point number. The lattice structure and the evolution rule remain the same. One important improvement to enhance the computational efficiency has been made for the LBM was that the linearization of collision operator [25]. The uniform lattice structure was remaining unchanged.

The starting point in the lattice Boltzmann scheme is by tracking the evolution of the single-particle distribution function. The concept of particle distribution has already well developed in the field of statistical mechanics while discussing the kinetic theory of gases and liquids [26]. The definition implies the probable number of molecules in a certain volume at a certain time made from a huge number of particles in a system that travel freely, without collision, for distance (mean free path) long compared to their sizes. Once the distribution functions are obtained, the hydrodynamics equations can be derived.

Although LBM approach treats gases and liquids as systems consisting of individual particles, the primary goal of this approach is to build a bridge between the mesoscopic and macroscopic dynamics, rather than to deal with macroscopic dynamics directly. In other words, the goal is to derive macroscopic equations from mesoscopic dynamics by means of statistic, rather than to solve macroscopic equations.

The LBM has a number of advantages over other conventional computational fluid dynamics approaches. The algorithm is simple and can be implemented with a kernel of just a few hundred lines [27]. The algorithm can also be easily modified to allow for the application of other, more complex simulation components. For example, the LBM can be extended to describe the evolution of binary mixtures, or extended to allow for more complex boundary conditions [28-30]. Thus the LBM is an ideal tool in fluid simulation.

The objective of present paper is to introduce and discuss the formulation of LBM in simulating fluid flow problem. The derivation of macroscopic continuity and momentum equations from mesoscale Boltzmann equation is discussed in details. After showing how the formulation of LBM fits in to the framework of macroscopic flow, numerical results of lid-driven flow in various types of cavity are presented to highlight the applicability of the approach.

This paper is arranged as follow. In the next section, we begin with the derivation of lattice Boltzmann BGK equation from the continuous classical Boltzmann equation. The complicated collision integral is replaced with the equilibrium function to simplify the collision process. We then demonstrate the capability of the present approach in formulating macroscopic dynamics equation of fluid flow using multiscale time approach (Chapman-Enkogs expansion). The simulation results of lid-driven flow in cavity are shown and discussed in section three. The fluid flow behavior is expressed in terms of streamline, size and number of vortices, which are formed in the cavity. The final section concludes current study.

2. Mesoscopic Lattice Boltzmann Model

Ludwig Boltzmann (1844-1906) introduced a transport equation based on statistical mechanics describing the evolution of gas particle in a system as

\[
\frac{d\phi}{dt} + c \frac{d\phi}{dx} + a \frac{d\phi}{dc} = \Omega
\]  

(1)

where \( \phi \), \( c \), \( a \) and \( \Omega \) stand for density distribution function, mesoscopic speed, acceleration due to external force and collision function respectively. If there is no external force, Eq. (1) is no more than a hyperbolic wave equation with source term given as

\[
\frac{d\phi}{dt} + c \frac{d\phi}{dx} = \Omega
\]  

(2)

Any solution of the Boltzmann equation, Eq. (2), requires an expression for the collision operator \( \Omega \). If the collision is to conserve mass, momentum and energy, it is required that

\[
\int_{c} \left[ \frac{1}{c^4} \right] \Omega dc = 0
\]  

(3)

However, the expression for \( \Omega \) is too complex to be solved. Even if we only consider two-body collision, the collision integral term needs to consider the scattering angle of the binary collision, the speed and direction before and after the collision, etc. Any replacement of collision must satisfy the conservation law as expressed in Eq. (3). The idea behind this replacement is that large amount of detail of two-body interaction is not likely to influence significantly the values of many experimental measured quantities [31]. There are a few version of collision operator published in the literature. However, the most well accepted version due to its simplicity and efficiency is the Bhatnagar-Gross-Krook collision model with a single relaxation time. The equation that represents this model is given by

\[
\Omega(f) = -\frac{f-f^eq}{\tau}
\]  

(4)

where \( f^eq \) is the equilibrium distribution function and \( \tau \) is the time to reach equilibrium condition during collision process and is often called the relaxation time. Eq. (4)
also describes that $1/\tau$ of non-equilibrium distribution relaxes to equilibrium state within time $\tau$ on every collision process. Substituting Eq. (4) into Eq. (2) gives

$$\frac{\partial f}{\partial t} + \mathbf{c} \frac{\partial f}{\partial x} = \frac{f - f_{eq}}{\tau}$$

which is known as the Boltzmann BGK equation.

Eq. (5) describes two main processes at mesoscale level. The left hand side refers to the propagation of distribution function to the next node in the direction of its probable velocity, and the right hand side represents the collision of the particle distribution functions. In lattice Boltzmann formulation, magnitude of $\mathbf{c}$ is set up so that in each time step $\Delta t$, every distribution function propagates in a distance of lattice nodes spacing $\Delta x$. This will ensure that distribution function arrives exactly at the lattice nodes after $\Delta t$ and collides simultaneously.

In order to apply Eq. (5) into the digital computer, the mesoscopic velocity space has to be discretised. This can be done by discretising the physical space into uniform lattice nodes. Every node in the network is then connected with its neighbours through a number of lattice velocities to be determined through the model chosen. The general form of the lattice velocity model is expressed as $\text{DnQm}$ where $D$ represents spatial dimension and $Q$ is the number of connection (lattice velocity) at every node. There are many lattice velocity models published in the literature, however, the most well used due to its simplicity is $\text{D2Q9}$ and shown in Figure 1.

![Figure 1 D2Q9 lattice Model](image)

After discretisation in velocity space, Eq. (5) can be rewritten in the following form

$$\frac{\partial f_{i\alpha}}{\partial t} + \mathbf{c}_{i\alpha} \frac{\partial f_{i\alpha}}{\partial x} = \frac{f_{i\alpha} - f_{eq}^{i\alpha}}{\tau}$$

where $i$ refers to the number of discrete velocity, $\sigma$ refers to the direction of mesoscopic velocity where $\sigma = 0$ when $i = 0$, $\sigma = 1$ when $i = 1,3,5,7$ and $\sigma = 2$ when $i = 2,4,6,8$. The Eulerian expression of the left hand side of Eq. (6) can be transformed into the Lagrangian form. To do this, we take Euler time step in conjunction with an upwind spatial discretization and then setting the grid spacing divided by the time step equal to the velocity. This leads to the well-known lattice Boltzmann BGK equation

$$f_{i\alpha}(\mathbf{x} + \mathbf{c}_{i\alpha} \varepsilon, t + \varepsilon) - f_{i\alpha}(\mathbf{x}, t) = - \frac{f_{i\alpha} - f_{eq}^{i\alpha}}{\tau}$$

where $\varepsilon$ is a small lattice time unit in physical unit.

### 2.1 Derivation of Macroscopic Equations

The equilibrium distribution function $f_{eq}$ is chosen so that we can reconstruct the hydrodynamics of fluid flow. The general form of $f_{eq}$ can be written as

$$f_{i\alpha}^{eq} = A_{\sigma} + B_{\sigma}(c_{i\alpha} \cdot \mathbf{u}) + C_{\sigma}(c_{i\alpha} \cdot \mathbf{u})^2 + D_{\sigma} \mathbf{u}^2$$

Here $A_{\sigma}$, $B_{\sigma}$, $C_{\sigma}$ and $D_{\sigma}$ are the coefficients to be determined based of Chapmann-Enskog procedure. For the rest particle, Eq. (8) becomes

$$f_{i0}^{eq} = A_{0} + D_{0} \mathbf{u}^2$$

and

$$f_{i1}^{eq} = A_{1} + B_{1}(c_{i1} \cdot \mathbf{u}) + C_{1}(c_{i1} \cdot \mathbf{u})^2 + D_{1} \mathbf{u}^2$$

$$f_{i2}^{eq} = A_{2} + B_{2}(c_{i2} \cdot \mathbf{u}) + C_{2}(c_{i2} \cdot \mathbf{u})^2 + D_{2} \mathbf{u}^2$$

for other particles. This gives

$$B_{0} = C_{0} = 0$$

The symmetric properties of the tensor $\sum_i c_{i\sigma a} c_{i\beta b} \cdots$ are needed in the derivation and given as follow:

- The odd orders of tensor are equal to zero.
- The second order tensor satisfies $\sum_i c_{i\sigma a} c_{i\beta b} = 2c_{\sigma\beta}^2 \delta_{\alpha\beta}$ where $\delta_{\alpha\beta}$ is the Kronecker delta and $c_1 = 1$ and $c_1 = \sqrt{2}$.
- The fourth order tensor has an expression as $\sum_i c_{i\sigma a} c_{i\alpha b} c_{i\gamma c} c_{i\delta d}$ for $\sigma = 1$ and $\sum_i c_{i\sigma a} c_{i\alpha b} c_{i\gamma c} c_{i\delta d} = 4\Delta_{\sigma\alpha\beta\gamma\delta}$ where $\Delta_{\alpha\beta\gamma\delta} = \delta_{\sigma\alpha} \delta_{\beta\gamma} + \delta_{\alpha\beta} \delta_{\gamma\delta} + \delta_{\alpha\beta} \delta_{\gamma\delta}$.

By considering conservation laws of $\sum_{\sigma} \sum_{i\alpha} f_{i\alpha}^{eq} = \rho$ and $\sum_{\sigma} \sum_{i\alpha} f_{i\alpha} c_{i\alpha} = \rho \mathbf{u}$ results in

$$\sum_{\sigma} \sum_{i\alpha} f_{i\alpha}^{eq} = A_0 + 4A_1 + 4A_2 + (2C_1 + 4C_2 + D_0 + 4D_1 + 4D_2) \mathbf{u}^2 = \rho$$

$$\sum_{\sigma} \sum_{i\alpha} f_{i\alpha} c_{i\alpha} = (2B_1 + 4B_2) \mathbf{u} = \rho \mathbf{u}$$
These give constraints for coefficients $A_{\sigma}$, $B_{\sigma}$, $C_{\sigma}$ and $D_{\sigma}$ as

\begin{align}
A_0 + 4A_1 + 4A_2 &= \rho \\
2C_1 + 4C_2 + D_0 + 4D_1 + 4D_2 &= 0
\end{align}

and

\begin{align}
2B_1 + 4B_2 &= \rho
\end{align}

To satisfy Eq. (15) we chose $A_0 = \frac{4}{5} \rho$, $A_1 = \frac{1}{5} \rho$, $A_2 = \frac{1}{36} \rho$.

We next decompose the timescale into slow and fast timescale. This is to represent two different phenomena occur at different timescale such as advection and diffusion.

\begin{align}
\frac{\partial}{\partial t} = \frac{\partial}{\partial t_0} + \epsilon \frac{\partial}{\partial t_1}
\end{align}

where $\epsilon$ plays the role of Knudsen number [32]. We also expand $f$ about $f_{eq}$,

\begin{align}
f_{\sigma i} = f_{\sigma i}^{eq} + \epsilon f_{\sigma i}^1 + \epsilon^2 f_{\sigma i}^2 + O(\epsilon^3)
\end{align}

Here $\Sigma_{\sigma} \Sigma_{\sigma i} f_{\sigma i}^{(n)} = 0$ and $\Sigma_{\sigma} \Sigma_{\sigma i} f_{\sigma i}^{(n)} c_{\sigma i} = 0$ to imply that the non-equilibrium distributions do not contribute to the local values of density and momentum.

Taylor expanding of Eq. (7) and retaining terms up to $O(\epsilon^2)$ results in

\begin{align}
\epsilon \left[ \frac{\partial}{\partial t} + (c_{\sigma i} \cdot \nabla) \right] f_{\sigma i} + \frac{\epsilon^2}{2} \frac{\partial}{\partial t} \left( c_{\sigma i} \cdot \nabla \right) f_{\sigma i} + O(\epsilon^3) = -\frac{f_{\sigma i} - f_{\sigma i}^{eq}}{\tau}
\end{align}

Substituting Eqs. (18) and (19) into Eq. (20), the equation to order of $\epsilon$ and $\epsilon^2$ are

\begin{align}
\left( \frac{\partial}{\partial t_0} + (c_{\sigma i} \cdot \nabla) \right) f_{\sigma i}^{eq} = -\frac{1}{\tau} f_{\sigma i}^1
\end{align}

and

\begin{align}
\left( \frac{\partial}{\partial t_0} + (c_{\sigma i} \cdot \nabla) \right) f_{\sigma i}^1 + \left( \frac{\partial}{\partial t_0} + (c_{\sigma i} \cdot \nabla) \right) f_{\sigma i}^{eq} + \frac{\partial f_{\sigma i}^{eq}}{\partial t_1} = -\frac{1}{\tau} f_{\sigma i}^2
\end{align}

respectively. Eq. (22) can be further simplified by using Eq. (21) gives

\begin{align}
\frac{\partial f_{\sigma i}^{eq}}{\partial t_1} + \left( \frac{\partial}{\partial t_0} + (c_{\sigma i} \cdot \nabla) \right) \left( 1 - \frac{1}{\tau_1} \right) f_{\sigma i}^1 = -\frac{1}{\tau} f_{\sigma i}^2
\end{align}

The first order continuity equation can be obtained by taking summation of Eq. (21) respect to $\sigma$ and $i$ as

\begin{align}
\frac{\partial p}{\partial t_0} + \nabla \cdot (\rho \mathbf{u}) = 0
\end{align}

Taking the same summation of Eq. (23) gives

\begin{align}
\frac{\partial p}{\partial t_1} = 0
\end{align}

Combining Eq. (24) and (25) gives the correct form of the continuity equation

\begin{align}
\frac{\partial p}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0
\end{align}

We next multiply Eq. (24) with $c_{\sigma i}$ and taking the summation as above gives

\begin{align}
\frac{\partial}{\partial t_0} (\rho \mathbf{u} + \nabla \cdot \Pi^{eq}) = 0
\end{align}

where $\Pi^{eq} = \Sigma_{\sigma} \Sigma_{\sigma i} (c_{\sigma i} c_{\sigma i}) f_{\sigma i}^{eq}$ is the momentum flux tensor. Substituting the expression of the equilibrium distribution, $\Pi^{eq}$ can be written as

\begin{align}
\Pi_{\sigma \beta}^{eq} = [2A_1 + 4A_2 + (4C_2 + 2D_1 + 4D_2)u^2] \delta_{\sigma \beta} + 8C_2 u_{\alpha} u_{\beta} + (2C_1 - 8C_2) u_{\alpha} u_{\beta} \delta_{\alpha \beta}
\end{align}

which are the pressure term and two nonlinear terms. This gives

\begin{align}
8C_2 = \rho
\end{align}

and

\begin{align}
2A_1 + 4A_2 = c_2^2 \rho
\end{align}

where $c_2^2 = 1/3$ is speed of sound. In order to obtain a velocity independent pressure and Galilean invariance, we choose

\begin{align}
4C_2 + 2D_1 + 4D_2 = 0
\end{align}

and

\begin{align}
2C_1 - 8C_2 = 0
\end{align}

This gives the final expression for $\Pi^{eq}$ as

\begin{align}
\Pi_{\sigma \beta}^{eq} = c_2^2 \rho \delta_{\alpha \beta} + p u_{\alpha} u_{\beta}
\end{align}

Substituting Eq. (33) into Eq. (27) results in Euler equation as

\begin{align}
\frac{\partial}{\partial t_0} (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \mathbf{u}) = -\nabla (c_2^2 \rho)
\end{align}

and the pressure is given by $p = c_2^2 \rho$. 

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We next multiply Eq. (25) with \( c_{i10} \) and taking the summation respect to \( \sigma \) and \( i \)

\[
\frac{\partial}{\partial t} (\rho u) + \nabla \cdot (1 - \frac{1}{2\pi}) \Pi^1 = 0
\]  
(35)

Substituting the expression of the non-equilibrium distribution and using Eqs. (26) and (33) leads to

\[
\Pi^1_{ab} = -\tau \left[ c_2^2 \partial_{\sigma} \partial_{\sigma} (\rho u_{ua} + \frac{\partial}{\partial \sigma} (\rho u_{ua} u_{ub})) + \partial_{\sigma} (2B_1 - 8B_2) u_{a\sigma} u_{b\sigma} + 4\partial_{\sigma} (B_2 u_{i\alpha}) \delta_{ab} + 4\partial_{\sigma} (B_1 u_{i\alpha}) + 4\partial_{\sigma} (B_2 u_{i\alpha}) \right] 
\]

(36)

To maintain isotropy, we set

\[
2B_1 - 8B_2 = 0
\]

(37)

Recalling Eq. (17), gives the expression for \( B_1 \) and \( B_2 \)

\[
B_1 = \frac{\rho}{12}, \quad B_2 = \frac{\rho}{12}
\]

(38)

Using Eq. (34) in the form

\[
\frac{\partial}{\partial \sigma} (\rho u_{ua} u_{ub}) = -u_a \partial_{\sigma} (c_2^2 \rho) - u_{ub} \partial_{\sigma} (c_2^2 \rho) - \partial_{\sigma} (\rho u_{ua} u_{ub}) \]

(39)

Eq. (36) can be written as

\[
\Pi^1_{ab} = -\tau \left[ \left( \frac{1}{3} - c_2^2 \right) \partial_{\sigma} (\rho u_{ua} + \frac{\partial}{\partial \sigma} (\rho u_{ua} u_{ub})) + \frac{1}{3} \partial_{\sigma} (\rho u_{ua} u_{ub}) - u_{a\sigma} \partial_{\sigma} (c_2^2 \rho) - u_{b\sigma} \partial_{\sigma} (c_2^2 \rho) - \partial_{\sigma} (\rho u_{ua} u_{ub}) \right] 
\]

(40)

Combining Eqs. (34), (35) and (39) gives the momentum equation in two dimensions

\[
\frac{\partial}{\partial t} (\rho u_{ua}) + \partial_{\sigma} (\rho u_{ua} u_{ub}) = -\partial_{\sigma} p + \partial_{\sigma} \left[ \mu \left( \partial_{\sigma} u_{ua} + \partial_{\sigma} u_{ub} - \frac{2}{3} u_{a\sigma} \delta_{ab} \right) \right] 
\]

(41)

where

\[
\mu = \rho \left( \frac{2\pi - 1}{\pi} \right)
\]

(42)

For an incompressible fluid, the momentum equation becomes

\[
\frac{\partial}{\partial t} (\rho u_{ua}) + \partial_{\sigma} (\rho u_{ua} u_{ub}) = -\partial_{\sigma} p + \partial_{\sigma} \left[ \mu \left( \partial_{\sigma} u_{ua} + \partial_{\sigma} u_{ub} - \frac{2}{3} u_{a\sigma} \delta_{ab} \right) \right]
\]

(43)

The remaining coefficients are determined as follow

\[
A_0 = \frac{4}{3} \rho, \quad B_0 = 0, \quad C_0 = 0, \quad D_0 = -\frac{2}{3} \rho \]

\[
A_1 = \frac{1}{3} \rho, \quad B_1 = \frac{1}{3} \rho, \quad C_1 = \frac{1}{2} \rho, \quad D_1 = -\frac{1}{2} \rho \]

\[
A_2 = \frac{1}{36} \rho, \quad B_2 = \frac{1}{12} \rho, \quad C_2 = \frac{1}{8} \rho, \quad D_2 = -\frac{1}{24} \rho
\]

(44)

Finally, the equilibrium distribution functions can be written as follow

\[
f_{i0}^{eq} = \frac{4}{9} \rho \left[ 1 - \frac{3}{2} u_i^2 \right] \text{ for } i = 0
\]

(45)

\[
f_{i1}^{eq} = \frac{1}{9} \rho \left[ 1 + 3(c_{i1} \cdot u) + \frac{9}{2} (c_{i1} \cdot u)^2 - \frac{3}{2} u_i^2 \right] \text{ for } i = 1,3,5,7
\]

(46)

and

\[
f_{i2}^{eq} = \frac{1}{36} \rho \left[ 1 + 3(c_{i1} \cdot u) + \frac{9}{2} (c_{i1} \cdot u)^2 - \frac{3}{2} u_i^2 \right] \text{ for } i = 2,4,6,8
\]

(47)

From the above derivation, we can see that the mesoscale Boltzmann equation can lead to the macroscopic Navier-Stokes equation by the Chapman Enskog expansion.

I. Numerical Investigation of Lid-Driven Cavity Flow

Lid-driven cavity flow is a phenomenon where the fluid is set into motion by a part of the containing boundary. This type of flow is crucial for analyzing fundamental aspects of recirculating fluids: in spite of the apparently simple geometry, lid-driven cavity flows may involve high degree of complexity.

In this paper, we firstly demonstrate the numerical results obtained by mesoscale lattice Boltzmann computation for lid-driven flow in a square cavity at various values of Reynolds numbers. The benchmark numerical solutions based from macroscale Navier-Stokes formulation are brought for the sake of validation. We then extend our mesoscale computation for lid-driven flow in isosceles triangular cavities. The computational results will be demonstrated based on the plot of streamlines to predict the behavior of vortices inside the respective cavity.

3.1 Mesoscale Code Validations

Figure 2(a) – (g) shows plots of streamline for the Reynolds numbers considered. They are apparent that the flow structures are in good agreement with the results published in the literature by previous researchers [33-40]. For low Reynolds number (Re = 100), the center of the vortex is located at about one-third of the cavity depth from the top. As we increase the Reynolds numbers, the primary vortex moves towards the cavity center and increasing circular. In addition to the primary, a pair of counter rotating eddies develop at the lower corners of the cavity. At Re = 3200, a third secondary vortex is evolved in the upper left corner of the cavity. As we further increase the value of Reynolds number, the size of the secondary vortices become larger.

We next computed the location of the center of the main vortex for every Reynolds numbers and compared with the results in the literature [33,37]. The results are shown in Table 1.
Table 1 Location of the center of the main vortex for square cavity

<table>
<thead>
<tr>
<th>Re</th>
<th>Ghia et al. [33]</th>
<th>Vanka et al. [37]</th>
<th>LBM</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>(0.6172,0.7344)</td>
<td>(0.6188,0.7375)</td>
<td>(0.6200,0.7400)</td>
</tr>
<tr>
<td>400</td>
<td>(0.5547,0.6055)</td>
<td>(0.5563,0.6000)</td>
<td>(0.5600,0.6000)</td>
</tr>
<tr>
<td>1000</td>
<td>(0.5313,0.5625)</td>
<td>(0.5438,0.5625)</td>
<td>(0.5390,0.5650)</td>
</tr>
<tr>
<td>3200</td>
<td>0.5165,0.5469</td>
<td>-</td>
<td>0.5200,0.5400</td>
</tr>
<tr>
<td>5000</td>
<td>(0.5117,0.5352)</td>
<td>(0.5125,0.5313)</td>
<td>(0.5150,0.5350)</td>
</tr>
<tr>
<td>7500</td>
<td>(0.5117,0.5322)</td>
<td>-</td>
<td>(0.5150,0.5325)</td>
</tr>
<tr>
<td>10000</td>
<td>(0.5117,0.5333)</td>
<td>-</td>
<td>(0.5133,0.5283)</td>
</tr>
</tbody>
</table>

From the results presented in Table 1, we can see that the mesoscale LBM is able to produce an excellent agreement with the results predicted by conventional macroscale numerical methods.

![Streamline plots for square cavity for different Reynolds numbers](image)

**Figure 2** Streamline plots for square cavity for different Reynolds numbers

The velocity components along the vertical and horizontal lines through the cavity center together with the benchmark solutions are shown in Figure 3(a) – (g). Good agreement between the mesoscale LBM and the benchmark solutions are observed. It is noted that, the mesoscale LBM is able to capture the critical points in the tested problem.

![Velocity components along the vertical and horizontal lines through the cavity center](image)

**Figure 3** Velocity components along the vertical and horizontal lines through the cavity center (lines: LBM, symbol: Ghia et al [33])

3.2 Lid-driven flow in triangular cavity

In this sub section, we demonstrate flow behavior at different Reynolds numbers in a triangular cavity using plots of streamline in the system. We first considered isosceles triangle with 90° at bottom corner angle. The plots revealed two significant features. Firstly, as Reynolds numbers is increased, the primary vortex moves downstream to the right. For example, at low Reynolds number (Re = 100), the location of primary vortex is located about one-third from the top horizontal wall as shown in Figure 4. Apart from the primary vortex, no other vortex is visible for this value of Reynolds number. However, for Re = 400, the secondary vortex can be seen and located near the stagnant corner of the triangle. The shape of this secondary vortex becomes larger as Reynolds numbers is further increased up to Re = 1000 as shown in Figure 4(d).

The second significant feature is the number of vortices in the cavity which is increased as the Reynolds number is increased. Start from one main vortex for low Reynolds number, Figure 4 shows that the secondary vortex appears at Re = 400 and its size becomes larger and larger as we increase the Reynolds number. The tertiary vortex appears at the bottom tip of the cavity at the simulation of Reynolds number 3000. As we further increase the Reynolds number, a few vortices appear at the left and right diagonal boundaries of the cavity.

![Streamline plots for triangular cavity](image)
Figure 4 Streamline plots for isosceles right triangle with 90° being at the corner angle.

Table 2 Location of the center of the main vortex for isosceles triangle with 90° at corner angle

<table>
<thead>
<tr>
<th>Re</th>
<th>Location of primary vortex</th>
<th>X</th>
<th>Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td></td>
<td>0.5450</td>
<td>0.7600</td>
</tr>
<tr>
<td>400</td>
<td></td>
<td>0.6100</td>
<td>0.7500</td>
</tr>
<tr>
<td>700</td>
<td></td>
<td>0.5950</td>
<td>0.7150</td>
</tr>
<tr>
<td>1000</td>
<td></td>
<td>0.5875</td>
<td>0.7150</td>
</tr>
<tr>
<td>3000</td>
<td></td>
<td>0.7400</td>
<td>0.8200</td>
</tr>
<tr>
<td>5000</td>
<td></td>
<td>0.7350</td>
<td>0.8100</td>
</tr>
<tr>
<td>7000</td>
<td></td>
<td>0.7583</td>
<td>0.6200</td>
</tr>
<tr>
<td>10000</td>
<td></td>
<td>0.4117</td>
<td>0.5375</td>
</tr>
</tbody>
</table>

Figure 5 Streamline plots for 90° right corner triangular cavity.

Table 3 Location of the center of the main vortex.

<table>
<thead>
<tr>
<th>Re</th>
<th>Erturk and Gokcol [41]</th>
<th>LBM</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>(0.7090,0.8320)</td>
<td>(0.7100,0.8300)</td>
</tr>
<tr>
<td>500</td>
<td>(0.7070,0.7676)</td>
<td>(0.7100,0.7650)</td>
</tr>
<tr>
<td>1000</td>
<td>(0.6992,0.7559)</td>
<td>(0.7000,0.7550)</td>
</tr>
<tr>
<td>1500</td>
<td>-</td>
<td>(0.7000,0.7467)</td>
</tr>
<tr>
<td>2000</td>
<td>-</td>
<td>(0.7000,0.7467)</td>
</tr>
<tr>
<td>2500</td>
<td>(0.6973,0.7441)</td>
<td>(0.7000,0.7433)</td>
</tr>
</tbody>
</table>

Table 2 shows the computed center of primary vortex for every simulated Reynolds numbers. To the best of authors’ knowledge, such information on the coordinate of primary vortex at steady state has not been reported. Therefore, no comparison can be made to verify the computed results.

We then considered two types of an isosceles right triangle. They are 90° at right corner and 90° at left corner of the triangular cavities. The flow in these considered triangles is predicted up to Re = 2500. Figures 5 and 6 show the streamline plot for every simulated case. In Tables 3 and 4, we computed the location of the center of primary eddy and compared with Erturk and Gokcol [41] and found very good agreement. From the results predicted above, we can say that the flow behaves very different as we increase the Reynolds number and greatly affected by the shape of the geometry. In addition to that, these also demonstrate the capability of the current mesoscale method in predicting complex fluid flow behavior in this complex system.
II. Summary and Concluding Remarks

In this paper, a mesoscale numerical method, the lattice Boltzmann scheme was used to investigate the steady fluid flow behavior in a lid-driven cavity. We have shown the capability of lattice Boltzmann scheme to derive the macroscopic dynamics of fluid flow equations via the multiscale expansion. This led the current scheme to produce results with second order accuracy in space and time. The flow structure in square and triangular cavities has been successfully reproduced and compared with the benchmark solution available in the literature. Excellent agreement has been found. This gives us confidence to apply the current method to investigate other various types of fluid flow problems.

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Nor Azwadi C. Sidik was born in Kelantan on 23rd September 1977. He received Ph.D degree (2010) from Keio University, Japan. His current interest includes computational fluid dynamics, numerical methods and fluid structure interaction. Dr. Nor Azwadi is a senior lecturer at Department of Thermofluid, Universiti Teknologi Malaysia, Malaysia.

Fudhail Abdul Munir was born in Malaysia on 30th April 1982. He received his M.Sc. degree (2010) from University Teknologi Malaysia in Mechanical Engineering. He is currently a lecturer at the Department of Automotive in Universiti Technical Malaysia Malacca (UTeM). His current interest includes computational fluid dynamics, turbulent flows modeling and solid-fluid interaction analysis.