Chapter One:

Boundary Layer Theory

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1.1 Introduction
The concept of boundary layer was first introduced by a German engineer, Prandtl in 1904. According to Prandtl theory, when a real fluid flows past a stationary solid boundary, the flow will be divided into two regions.

i) A thin layer adjoining the solid boundary where the viscous force and rotation cannot be neglected.

ii) An outer region where the viscous force is very small and can be neglected. The flow behaviour is similar to the upstream flow.

1.2 Boundary layer definitions and characteristics
Consider a flow over a flat plate aligned in the direction of the flow as shown in Figure 1.1

![Flow over a flat horizontal surface](image)

Figure 1.1 Flow over a flat horizontal surface

The flow region can be divided into two layers

i) $0 \leq y \leq \delta$ inner layer/boundary layer flow where the viscous force effect is significant. Due to no-slip condition at the boundary surface, the first layer of fluid undergoes retardation. This retarded layer causes further retardation for the adjacent layer, thereby developing a thin region where the flow velocity increases from zero at the solid boundary and approaches the velocity of the main stream.
Due to the presence of velocity gradient inside the boundary layer region, the fluid particle begins to distort at the top of particle which has a larger speed than its bottom. This force causes the fluid particle begins to rotate as it enters the boundary layer region (See Figure 1.2). Therefore, this layer of fluid is also known as rotational flow.

ii) \( y > \delta \) Outer flow region where the viscous force is very small and can be neglected. There is no velocity gradient in this layer and the fluid particle will not rotate as it enters the outer region of flow. Therefore, the flow is also known as irrotational flow.

As shown by the figure, the boundary layer conditions are that the fluid sticks to the solid boundary

\[
 u = v = 0 \text{ on } y = 0
\]  

(1.1)

and at the outside of the boundary layer, the flow velocity is equal to the free stream velocity, that is

\[
 u = U \text{ on } y = \delta
\]  

(1.2)

The following boundary condition is also true for boundary layer flow

\[
 \frac{\partial u}{\partial y} = 0 \text{ when } y \geq \delta
\]  

(1.3)

indicates that the velocity distribution is uniform in \( y \)-direction outside the boundary layer.
1.2.1 Boundary layer thickness, $\delta$

The boundary layer thickness is defined as the *vertical distance from a flat plate to a point where the flow velocity reaches 99 per cent of the velocity of the free stream*.

The other definitions of boundary layer are the

- Boundary layer displacement thickness, $\delta^*$
- Boundary layer momentum thickness, $\theta$

1.2.2 Boundary layer displacement thickness, $\delta^*$

Consider two types of fluid flow past a stationary horizontal plate with velocity $U$ as shown in Figure 1.3.

Since there is no viscosity for the case of ideal fluid (Figure 1.3a), a uniform velocity profile is developed above the solid wall. However, for the case of real fluid with the presence of viscosity and no-slip on the wall, the velocity gradient is developed in the boundary layer region as shown in Figure 1.3b.

The velocity deficits through the element strip of cross section $b-b$ is $U - u$. Then the reduction of mass flow rate is obtained as
\[ \rho (U - u)bdy \] (1.4)

where \( b \) is the plate width. The total mass reduction due to the presence of viscosity compared to the case of ideal fluid

\[ \int_{0}^{\delta} \rho (U - u)bdy \] (1.5)

However, if we displace the plate upward by a distance \( \delta^* \) in section a-a to give mass reduction of \( \rho Ub \delta^* \), then the deficit of flowrates for the both cases will be identical if

\[ \int_{0}^{\delta} \rho (U - u)bdy = \rho Ub \delta^* \] (1.6)

and

\[ \delta^* = \int_{0}^{\delta} \left( 1 - \frac{u}{U} \right) dy \] (1.7)

Here, \( \delta^* \) is known as the boundary layer displacement thickness.

The displacement thickness represents the vertical distance that the solid boundary must be displaced upward so that the ideal fluid has the same mass flowrate as the real fluid.

Figure 1.4 boundary layer displacement thickness
1.2.3 Boundary layer momentum thickness, $\theta$

Another definition of boundary layer thickness, the boundary layer momentum thickness $\theta$, is often used to predict the drag force on the object surface.

Referring to Figure 1.3, again the velocity deficit through the element strip of cross section $b$-$b$ contributes to deficit in momentum flux as

$$\rho u (U - u) b dy$$

Thus, the total momentum reductions

$$\int_{0}^{\delta} \rho u (U - u) b dy$$

However, if we displace the plate upward by a distance $\theta$ at section a-a to give momentum reduction of $\rho U^2 b \theta$, then the momentum deficit for the both cases will be identical if

$$\int_{0}^{\delta} \rho u (U - u) b dy = \rho U^2 b \theta$$

and

$$\theta = \int_{0}^{\delta} \frac{u}{U} \left(1 - \frac{u}{U}\right) dy$$

Here, $\theta$ is known as the boundary layer displacement thickness.

The momentum thickness represents the vertical distance that the solid boundary must be displaced upward so that the ideal fluid has the same mass momentum as real fluid.

**Question:**

1. If the velocity distribution in laminar boundary layer over flat plate is assumed to be given by first order polynomial $u = a + by$, where $a$ and $b$ are constant, determine
a) The ratio of displacement thickness to boundary layer thickness
b) The ratio of momentum thickness to boundary layer thickness

Solution

Velocity distribution: \( u = a + by \)

The following boundary conditions must be satisfied

i) At \( y = 0, \ u = 0 \)

\[ 0 = a + b(0) = a \]

ii) At \( y = \delta, \ u = U \)

\[ U = b\delta \text{ or } b = \frac{U}{\delta} \]

Hence form of the velocity distribution is

\[ u = \frac{U}{\delta} \ y \]

Next, derive the expression for \( \delta^*/\delta \) and \( \theta/\delta \)

\[ \delta^* = \int_0^\delta \left( 1 - \frac{u}{U} \right) dy = \int_0^\delta \left( 1 - \frac{y}{\delta} \right) dy = \left[ y - \frac{1}{2} \frac{y^2}{\delta} \right]_0^\delta = \left[ \frac{\delta}{2} \frac{\delta^2}{\delta} \right] = \left[ \frac{1}{2} \frac{\delta}{\delta} \right] = \frac{1}{2} \delta \]

\[ \therefore \ \frac{\delta^*}{\delta} = \frac{1}{2} \ (\text{Ans}) \]

\[ \theta = \int_0^\delta \frac{u}{U} \left( 1 - \frac{u}{U} \right) dy = \int_0^\delta \frac{y}{\delta} \left( 1 - \frac{y}{\delta} \right) dy = \left[ \frac{1}{2} \frac{y^2}{\delta} - \frac{1}{3} \frac{y^3}{\delta^2} \right]_0^\delta = \left[ \frac{1}{2} \frac{\delta}{\delta} - \frac{1}{3} \frac{\delta^2}{\delta^2} \right] = \frac{1}{6} \delta \]

\[ \therefore \ \frac{\theta}{\delta} = \frac{1}{6} \ (\text{Ans}) \]
2. Show that, if the velocity distribution in laminar boundary layer over flat plate is assumed to be given by second order polynomial, the velocity distribution can be expressed as follow

\[ \frac{u}{U} = 2 \left( \frac{y}{\delta} \right) - \left( \frac{y}{\delta} \right)^2 \]

**Solution**

Second order polynomial of velocity distribution is expressed as

\[ u = a + by + cy^2 \]

The following boundary conditions must be satisfied

i) At \( y = 0, \ u = 0 \)

\[ 0 = a + b(0) + c(0)^2 \therefore a = 0 \]

ii) At \( y = \delta, \ u = U \)

\[ U = b\delta + c\delta^2 \]

\[ \ldots \text{(i)} \]

iii) At \( y = \delta, \ \frac{du}{dy} = 0 \)

\[ \left( \frac{du}{dy} \right)_{y=\delta} = b + 2cy = b + 2c\delta = 0 \therefore b = -2c\delta \]

Substitute \( b = -2c\delta \) in (i), we get

\[ U = (-2c\delta)\delta + c\delta^2 = -c\delta^2 \text{ or} \]

\[ c = -\frac{U}{\delta^2} \]

\[ \therefore b = 2c\delta = 2\delta \left( \frac{U}{\delta^2} \right) = \frac{2U}{\delta} \]

Hence form of the velocity distribution is

\[ u = \frac{2U}{\delta} y - \frac{U}{\delta^2} y^2 \text{ or} \]
\[
\frac{u}{U} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2 \text{(Ans)}
\]

3. By assuming that the velocity gradient in laminar boundary layer over flat plate is in the form of second order polynomial, calculate
   
a) The displacement thickness
   b) The momentum thickness

**Solution**

Displacement thickness, \( \delta^* \):

\[
\delta^* = \int_0^\delta \left(1 - \frac{1}{U}\frac{u}{U}\right)dy = \int_0^\delta \left[1 - \left(1 - 2\frac{y}{\delta} + \frac{y^2}{\delta^2}\right)\right]dy = \int_0^\delta \left(1 - 2\frac{y}{\delta} + \frac{y^2}{\delta^2}\right)dy
\]

\[
= \left[y - \frac{2y^2}{2\delta} + \frac{y^3}{3\delta^2}\right]_0^\delta = (\delta - \frac{2\delta}{3}) = \frac{1}{3}\delta
\]

\[
\delta^* = \frac{1}{3}\delta \text{ (Ans)}
\]

Momentum thickness, \( \theta \):

\[
\theta = \int_0^\delta \frac{u}{U} \left(1 - \frac{1}{U}\frac{u}{U}\right)dy = \int_0^\delta \left(2\frac{y}{\delta} - \frac{y^2}{\delta^2}\right) \left(1 - \left(1 - \frac{y^2}{\delta^2}\right)\right)dy
\]

\[
= \int_0^\delta \left(2\frac{y}{\delta} - \frac{y^2}{\delta^2}\right) \left(1 - 2\frac{y}{\delta} + \frac{y^2}{\delta^2}\right)dy
\]

\[
= \int_0^\delta \left(2\frac{y}{\delta} - \frac{y^2}{\delta^2} + 2\frac{y^2}{\delta^2} - \frac{y^2}{\delta^2} + 2\frac{y^3}{\delta^3} - \frac{y^4}{\delta^4}\right)dy = \int_0^\delta \left(2\frac{y}{\delta} - \frac{5}{3}\frac{y^2}{\delta^2} + \frac{4}{3}\frac{y^3}{\delta^3} - \frac{y^4}{\delta^4}\right)dy
\]

\[
= \left[\frac{2}{2}\frac{y^2}{\delta} - \frac{5}{3}\frac{y^3}{\delta^2} + \frac{4}{3}\frac{y^4}{\delta^3} - \frac{1}{5}\frac{y^5}{\delta^4}\right]_0^\delta = \left(\delta - \frac{5}{3}\right) + \delta - \frac{1}{5}\delta = \frac{2}{15}\delta
\]

\[
\theta = \frac{2}{15}\delta \text{ (Ans)}
\]
4. If the velocity distribution in laminar boundary layer over flat plate is assumed to be

given by \( \frac{u}{U} = \sin \left( \frac{\pi y}{2 \delta} \right) \), determine

a) The ratio of momentum thickness to displacement thickness

b) The ratio of momentum thickness to boundary layer thickness

Solution

Velocity distribution: \( \frac{u}{U} = \sin \left( \frac{\pi y}{2 \delta} \right) \)

a) \( \frac{\theta}{\delta^*} \):

\[
\delta^* = \int_0^\delta \left( 1 - \frac{u}{U} \right) dy = \int_0^\delta \left( 1 - \sin \left( \frac{\pi y}{2 \delta} \right) \right) dy = \int_0^\delta \left( 1 - \sin \left( \frac{\pi y}{2 \delta} \right) \right) dy
\]

\[
= \left[ y + \frac{\cos \left( \frac{\pi y}{2 \delta} \right)}{\frac{\pi}{2 \delta}} \right]_0^\delta = (\delta + 0) - \left( 0 + \frac{2}{\pi} \right) = \left( \frac{\pi - 2}{\pi} \right) \delta
\]

\[
\theta = \int_0^\delta \frac{u}{U} \left( 1 - \frac{u}{U} \right) dy = \int_0^\delta \sin \left( \frac{\pi y}{2 \delta} \right) \left( 1 - \sin \left( \frac{\pi y}{2 \delta} \right) \right) dy
\]

\[
= \int_0^\delta \left[ \sin \left( \frac{\pi y}{2 \delta} \right) - \left( \sin \left( \frac{\pi y}{2 \delta} \right) \right)^2 \right] dy = \int_0^\delta \sin \left( \frac{\pi y}{2 \delta} \right) - \frac{1 - \cos \left( \frac{\pi y}{\delta} \right)}{2} dy
\]

\[
\left( \therefore \sin^2 \theta = \frac{1 - \cos 2\theta}{2} \right)
\]

\[
= \left[ -\frac{\cos \left( \frac{\pi y}{2 \delta} \right)}{\frac{\pi}{2 \delta}} - \frac{\sin \left( \frac{\pi y}{2 \delta} \right)}{2 \frac{\pi}{2 \delta}} \right]_0^\delta = \left[ 0 - \frac{\delta}{2} + 0 \right] - \left( - \frac{2}{\pi} + 0 + 0 \right) = \left( \frac{4 - \pi}{2\pi} \right) \delta
\]

\[
\therefore \frac{\theta}{\delta^*} = \left( \frac{4 - \pi}{2\pi} \right) \delta
\]

\[
\left( \frac{\pi - 2}{\pi} \right) \delta = \left( \frac{4 - \pi}{\pi - 2} \right) \times \left( \frac{\pi}{\pi - 2} \right) = 2 \left( \frac{4 - \pi}{\pi - 2} \right)
\]
\[ \frac{\theta}{\delta^*} = 2 \left( \frac{4 - \pi}{\pi - 2} \right) \text{(Ans)} \]

(b) \[ \frac{\theta}{\delta} : \]

\[ \frac{\theta}{\delta} = \left( \frac{4 - \pi}{2\pi} \right) \delta = \left( \frac{4 - \pi}{2\pi} \right) \]

\[ \frac{\theta}{\delta} = \left( \frac{4 - \pi}{2\pi} \right) \text{(Ans)} \]

5. The velocity distribution in the boundary layer is given by

\[ \frac{u}{U} = \left( \frac{y}{\delta} \right)^{\frac{5}{7}} \]

Calculate the displacement thickness and momentum thickness

**Solution**

Velocity distribution: \[ \frac{u}{U} = \left( \frac{y}{\delta} \right)^{\frac{5}{7}} \]

\[ \delta^* = \int_0^\delta \left( 1 - \frac{u}{U} \right) dy = \int_0^\delta \left( 1 - \left( \frac{y}{\delta} \right)^{\frac{5}{7}} \right) dy \]

\[ = \left[ y - \frac{7}{8} \left( \frac{y}{\delta} \right)^{\frac{7}{8}} \right]_0^\delta = \left( \delta - \frac{7}{8} \delta^{\frac{7}{8}} \right) = \left( \delta - \frac{7}{8} \delta \right) = \frac{1}{8} \delta \]

\[ \delta^* = \frac{1}{8} \delta \text{ (Ans)} \]

\[ \theta = \int_0^\delta \frac{u}{U} \left( 1 - \frac{u}{U} \right) dy = \int_0^\delta \left( \frac{y}{\delta} \right)^{\frac{5}{7}} \left( 1 - \left( \frac{y}{\delta} \right)^{\frac{5}{7}} \right) dy \]

\[ = \left[ \frac{7}{8} \delta^{\frac{7}{8}} - \frac{7}{9} \left( \frac{y}{\delta} \right)^{\frac{9}{8}} \right]_0^\delta = \left( \frac{7}{8} \delta - \frac{7}{9} \delta \right) = \frac{7}{72} \delta \]
\[ \theta = \frac{7}{72} \delta \quad (\text{Ans}) \]

6. The velocity distribution in the boundary layer over the surface of highway was observed to be

\[ \frac{u}{U} = \left( \frac{y}{\delta} \right)^{0.22} \]

The free stream velocity is 20 m/s and boundary layer thickness of 5 cm in a certain section. Calculate the displacement thickness and momentum thickness in the section under consideration.

**Solution**

Velocity distribution: \[ \frac{u}{U} = \left( \frac{y}{\delta} \right)^{0.22} \]

\( U = 20 \text{ m/s}; \ \delta = 5 \text{ cm} \)

The displacement thickness, \( \delta^* \)

\[
\delta^* = \int_0^\delta \left( 1 - \frac{u}{U} \right) \, dy = \int_0^\delta \left( 1 - \left( \frac{y}{\delta} \right)^{0.22} \right) \, dy
\]

\[
= \left[ y - \frac{1}{1.22} \left( \frac{y^{1.22}}{\delta^{0.22}} \right) \right]_0^\delta = \left( \delta - \frac{\delta}{1.22} \right) = \frac{0.22}{1.22} \delta = \frac{0.22}{1.22} \times 5
\]

\[
= 0.9016 \text{ cm (Ans)}
\]

\[
\theta = \int_0^\delta \frac{u}{U} \left( 1 - \frac{u}{U} \right) \, dy = \int_0^\delta \left( \frac{y}{\delta} \right)^{0.22} \left( 1 - \left( \frac{y}{\delta} \right)^{0.22} \right) \, dy
\]

\[
= \left[ \frac{1}{1.22} \left( \frac{y^{1.22}}{\delta^{0.22}} \right) - \frac{1}{1.44} \left( \frac{y^{1.44}}{\delta^{0.44}} \right) \right]_0^\delta = \left( \frac{\delta}{1.22} - \frac{\delta}{1.44} \right) = \frac{0.22}{1.7568} \delta = \frac{0.22}{1.7568} \times 5
\]

\[
= 0.6261 \text{ cm (Ans)}
\]
Expression for $\delta^*$ and $\theta$ using various types of velocity profiles in boundary layer is tabulated in Table 1.1

<table>
<thead>
<tr>
<th>Types of velocity distribution</th>
<th>Boundary layer displacement thickness, $\delta^*$</th>
<th>Boundary layer momentum thickness, $\theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear profile $u = \frac{y}{\delta}$</td>
<td>$\delta$</td>
<td>$\frac{\delta}{6}$</td>
</tr>
<tr>
<td>Parabolic profile, $\frac{u}{U} = \left( \frac{y}{\delta} \right)^2 + 2 \left( \frac{y}{\delta} \right)$</td>
<td>$\frac{\delta}{3}$</td>
<td>$\frac{2}{15} \delta$</td>
</tr>
<tr>
<td>Cubic profile, $\frac{u}{U} = \frac{3}{2} \left( \frac{y}{\delta} \right) - \frac{1}{2} \left( \frac{y}{\delta} \right)^3$</td>
<td>$\frac{3}{8} \delta$</td>
<td>$\frac{39}{280} \delta$</td>
</tr>
<tr>
<td>Sin-Cos profile, $\frac{u}{U} = \sin \left( \frac{\pi y}{2 \delta} \right)$</td>
<td>$\delta \left( 1 - \frac{2}{\pi} \right)$</td>
<td>$\delta \left( \frac{2}{\pi} - \frac{1}{2} \right)$</td>
</tr>
<tr>
<td>Turbulent profile, $\frac{u}{U} = \left( \frac{y}{\delta} \right)^{\frac{1}{2}}$</td>
<td>$\frac{\delta}{8}$</td>
<td>$\frac{7}{72} \delta$</td>
</tr>
</tbody>
</table>

1.3 Blassius Boundary Layer Solution

In 1908, H. Blassius, one of Prandtl student proposed simplified equations for the boundary layer flow by assuming that

$v << u \quad (1.12)$
Therefore, the fluid flow equations are reduced to

\[
\frac{\partial u}{\partial x} + \nu \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2}
\]  
(1.14)

From the Bernoulli equation, we know that

\[
\frac{1}{\rho} \frac{\partial p}{\partial x} = -U \frac{\partial U}{\partial x}
\]  
(1.15)

Then, the equation in the boundary layer becomes

\[
\frac{u}{\partial x} + v \frac{\partial u}{\partial y} = U \frac{\partial U}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2}
\]  
(1.16)

For the flow specifically over the flat surface, we can assume uniform horizontal velocity and therefore \( \partial U/\partial x = 0 \), thus

\[
\frac{u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2}
\]  
(1.17)

Here we introduce the equation of stream function as follow

\[
u = \frac{\partial \psi}{\partial y}, v = -\frac{\partial \psi}{\partial x}
\]  
(1.18)

Substitute back into the above equation gives

\[
\frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial y^2} = \nu \frac{\partial^3 \psi}{\partial y^3}
\]  
(1.19)

Blassius then introduced the dimensionless similarity variable \( \eta = (U/t)^{1/2} y \) and the stream function \( \psi = (t^2 U)^{1/2} f(\eta) \), where \( f(\eta) \) is an unknown function.

By using the rules of differentiation, we obtain as the following
\[
\frac{\partial \psi}{\partial x} = \frac{1}{2} \left( \frac{\nu U}{x} \right)^{3/2} \left[ f - \eta f' \right]
\]

\[
\frac{\partial \psi}{\partial y} = U f'
\]

\[
\frac{\partial^2 \psi}{\partial x \partial y} = -\frac{1}{2x} U \eta f''
\]

\[
\frac{\partial^3 \psi}{\partial y^3} = U \left( \frac{U}{\nu x} \right)^{1/2} f''
\]

\[
\frac{\partial^3 \psi}{\partial y^3} = U \left( \frac{U}{\nu x} \right)^{1/2} f''
\]

\[
(1.20)
\]

where \( f' = \partial f / \partial \eta \), \( f'' = \partial^2 f / \partial \eta^2 \), \( f''' = \partial^3 f / \partial \eta^3 \)

Substituting all the differentiation terms into Eq. (1.19) leads to

\[
f'' + \frac{1}{2} f f'' = 0
\]

(1.21)

Eq. (1.21) can be solved numerically by taking the boundary conditions as follow

\[
f(0) = 0
\]

\[
\frac{\partial f}{\partial \eta}(0) = 0
\]

\[
\frac{\partial^2 f}{\partial \eta^2} \to 0 \text{ when } \eta \to \infty
\]

(1.22)

The solution (Blassius solution) is tabulated as follow

| \( \eta = y(U/\nu x)^{1/2} \ | \ | f'(\eta) = u/U \ | \ \eta \ | \ f''(\eta) \ |
|---|---|---|---|
| 0 | 0 | 3.6 | 0.9233 |
| 0.4 | 0.1328 | 4.0 | 0.9555 |
| 0.8 | 0.2647 | 4.4 | 0.9759 |
| 1.2 | 0.3938 | 4.8 | 0.9878 |
| 1.6 | 0.5168 | 5.0 | 0.9916 |
From the solution, it is found that $\frac{u}{U} \approx 0.99$ when $\eta = 5$, thus, from the similarity variables

$$s = \left( \frac{U}{\nu x} \right)^{1/2} \delta$$

(1.23)

or

$$\delta = \frac{5x}{\text{Re}_{x}^{1/2}} \text{ where } \text{Re}_{x} = U/\nu x$$

(1.24)

Using some manipulations of algebra, the displacement and momentum thickness can be expressed as

$$\delta^{*} = \frac{1.72 \ln x}{\text{Re}_{x}^{1/2}}$$

(1.25)

and

$$\theta = \frac{0.664 x}{\text{Re}_{x}^{1/2}}$$

(1.26)

The wall shear stress is then determined by taking $\frac{\partial u}{\partial y}|_{y=0}$ and give

$$\tau_w = 0.332 U^{3/2} \sqrt{\frac{\rho \mu}{x}}$$

(1.27)

### 1.4. Von-Karmann Momentum Integral Equation

One of the main drawbacks of the Blassius solution is the limitation to the laminar flow over a flat surface only. In reality, most of the flows are turbulent. Therefore, the demand to replace Blassius solution with other equation which can predict the turbulent boundary layer
flow leads to a great work done by Von-Karmann in 1921. Von Karmann formulated a general equation from the conservation of momentum theory which can predict the boundary layer flow covers from laminar to turbulent regions. His equation contributes to advancement in the prediction of drag caused by shear forces on a body.

To see the formulation, consider a uniform flow past a flat plate and the fixed control volume as shown in Figure 1.5

![Figure 1.5 Control volume for the derivation of Von Karmann momentum integral equation](image)

We apply the horizontal component of the momentum equation. It gives

$$\sum F_x = \rho \int_1 uV \cdot n dA + \rho \int_2 uV \cdot n dA$$  \hspace{1cm} (1.28)

or

$$\sum F_x = \rho \int_1 U(-U) dA + \rho \int_2 u^2 dA$$  \hspace{1cm} (1.29)

$$\sum F_x = -\rho U^2 bh + \rho b \int_0^\delta u^2 dy$$  \hspace{1cm} (1.30)

Since there is no cross flow through streamline, the mass flow rate must be equal through section (1) and (2)
\[ U_{bh} = \int_{0}^{\delta} u \, dy \] (1.31)

which can be modified as

\[ \rho U^2 bh = \rho b \int_{0}^{\delta} U \, dy \] (1.32)

Substituting into Eq. (1.30)

\[ \sum F_x = -\rho b \int_{0}^{\delta} U \, dy + \rho \delta u^2 \, dy \] (1.33)

or

\[ \sum F_x = -\rho b \int_{0}^{\delta} u(U-u) \, dy \] (1.34)

We can see that Eq. (1.34) can be written in terms of the momentum thickness as follow

\[ \sum F_x = -\rho b U^2 \theta \] (1.35)

However, we know that the change of momentum contributes to the development of drag force on the solid surface such as

\[ \sum F_x = -D = -\int_{surface} \tau_{w} \, dA = -b \int_{surface} \tau_{w} \, dx \] (1.36)

This gives

\[ b \int \tau_{w} \, dx = \rho b U^2 \theta \] (1.37)

Differentiating both sides gives

\[ b \tau_{w} = \rho b U^2 \frac{d\theta}{dx} \] (1.38)

Now, we obtained the Von-Karmann momentum integral equation for the boundary layer flow over a flat plate

\[ \tau_{w} = \rho U^2 \frac{d\theta}{dx} \] (1.39)
Referring to Figure 1.6, the drag force on the solid surface can be determined by considering the drag force on small area $dA$ as

$$dF_D = \tau_w b dx$$

(1.40)

Then the total drag force on one side of plate with length $L$

$$F_D = \int dF_D = \int_0^L \tau_w b dx$$

(1.41)

It is also often convenient to use the dimensionless local friction coefficient $c_f$ and friction drag coefficient $c_D$ which are

$$c_f = \frac{\tau_w}{\frac{1}{2} \rho U^2}$$

(1.42)

and

$$c_D = \frac{F_D}{\frac{1}{2} \rho U^2 A} \quad \text{where } A: \text{total area (for this case, } A=bl)$$

(1.43)

Respectively.

The Blassius solution for these two coefficients is

$$c_f = \frac{0.664}{\text{Re}_{x}^{0.2}}$$

(1.44)

and
\[ c_D = \frac{1.328}{\text{Re}_L} \text{Re}^\frac{y_L}{z} \quad (1.45) \]

where \( \text{Re}_L = UL/\nu \)

Next we demonstrate the derivation of the equations of boundary layer thickness, boundary layer displacement thickness, boundary layer momentum thickness, local friction coefficient and total drag coefficient based on the following velocity distributions using the Von-Karmann momentum integral equation;

i) Linear velocity distribution

ii) Second order polynomial

iii) Third order polynomial

iv) Sin-cos profile

1.4.1 Linear velocity profile, \( \frac{u}{U} = \frac{y}{\delta} \)

\[ \tau_w = \rho U^2 \frac{d\theta}{dx} \quad \text{and} \quad \tau_w = \mu \frac{\partial u}{\partial y} \bigg|_{y=0} \quad (1.46) \]

Since \( \frac{\partial u}{\partial y} \bigg|_{y=0} = \frac{U}{\delta} \) therefore

\[ \rho U^2 \frac{d\theta}{dx} = \mu \frac{U}{\delta} \quad (1.47) \]

From the previous calculation, we know that \( \theta = \frac{\delta}{6} \), therefore

\[ \frac{\rho U^2}{6} \frac{d\delta}{dx} = \mu U \frac{\delta}{\delta} \quad (1.48) \]

or
\[ \delta \dot{d} \delta = \frac{6\mu}{\rho U} \, dx \]  
(1.49)

We integrate both sides and gives

\[ \frac{\delta^2}{2} = \frac{6\mu}{\rho U} \, x + C \]  
(1.50)

At \( x = 0, \delta = 0 \) and therefore \( C = 0 \)

\[ \delta^2 = \frac{12\mu x}{U} \]  
\[ = \frac{12\mu U x}{U x} \]  
\[ = \frac{12x^2}{U x / \nu} \]  
\[ = \frac{12x^2}{\text{Re}_x} \]  
(1.51)

Then we obtained the following

\[ \delta = \frac{3.464x}{\sqrt{\text{Re}_x}} \]  
(1.52)

The boundary layer displacement and momentum thickness are calculated as follow

\[ \delta^* = \frac{\delta}{2} = \frac{1.732x}{\sqrt{\text{Re}_x}} \]  
(1.53)

\[ \theta = \frac{\delta}{6} = \frac{0.5773x}{\sqrt{\text{Re}_x}} \]  
(1.54)

The wall shear stress is then calculated as

\[ \tau_w = \mu \, \frac{U}{\delta} = 0.2887 \frac{\mu U}{x} \sqrt{\text{Re}_x} \]  
(1.55)

The local wall friction coefficient is calculated as
\[ c_f = \frac{\tau_w}{\frac{1}{2} \rho U^2} = 0.2887 \frac{\mu U}{\rho U^2} \sqrt{\text{Re}_x} = 0.5774 \frac{\mu}{\rho U} \sqrt{\text{Re}_x} \] (1.56)

The wall friction coefficient is calculated as

\[ c_D = \frac{F_D}{\frac{1}{2} \rho U^2 A} = \frac{b(0.5774 \mu U \sqrt{\text{Re}_L})}{L} = 0.5774 \frac{\mu U}{\rho U} \sqrt{\text{Re}_L} \] (1.57)

where

\[ F_D = \int_0^L \tau_w b dx = \int_0^L 0.2887 \frac{\mu U}{\rho U} \sqrt{\text{Re}_x} b dx = \int_0^L 0.2887 \mu U \sqrt{\frac{U}{\rho U}} \frac{1}{2} b dx \] (1.58a)

\[ = b \left( 0.2887 \mu U \sqrt{\frac{U}{\rho U}} \right) \left[ \frac{x^{1/2}}{1/2} \right]_0^L = b \left( 0.5774 \mu U \sqrt{\frac{U}{\rho U}} \right) L^{1/2} = b \left( 0.5774 \mu U \sqrt{\text{Re}_L} \right) \] (1.58b)

Therefore

\[ c_D = \frac{b(0.5774 \mu U \sqrt{\text{Re}_L})}{\frac{1}{2} \rho U^2 b L} = 1.1548 \frac{\mu U}{\rho U} \sqrt{\text{Re}_L} = 1.1548 \frac{\sqrt{\text{Re}_L}}{\text{Re}_L} \] (1.59)
1.4.2 Second order polynomial \( \frac{u}{U} = \left( \frac{y}{\delta} \right)^2 + 2 \left( \frac{y}{\delta} \right) \)

\[
\tau_w = \rho U^2 \frac{d\theta}{dx} \text{ and } \tau_w = \mu \frac{\partial u}{\partial y}_{y=0} \tag{1.60}
\]

Since \( \frac{\partial u}{\partial y}_{y=0} = \frac{2U}{\delta} \) therefore

\[
\rho U^2 \frac{d\theta}{dx} = 2 \frac{\mu U}{\delta} \tag{1.61}
\]

From the previous calculation, we know that \( \theta = \frac{2\delta}{15} \), therefore

\[
\frac{2\rho U^2}{15} \frac{d\delta}{dx} = 2 \frac{\mu U}{\delta} \tag{1.62}
\]

or

\[
\delta \frac{d\delta}{dx} = \frac{15\mu}{\rho U} dx \tag{1.63}
\]

we integrate both sides and gives

\[
\frac{\delta^2}{2} = \frac{15\mu}{\rho U} x + C \tag{1.64}
\]

At \( x = 0, \delta = 0 \) and therefore \( C = 0 \)

\[
\delta^2 = \frac{30\mu x}{U} = \frac{30\mu x}{Ux} = \frac{30x^2}{Ux/U} = \frac{30x^2}{Re_x} \tag{1.65}
\]

Then we obtained the following
The boundary layer displacement and momentum thickness are calculated as follows:

\[ \delta = \frac{5.48 x}{\sqrt{\text{Re}_x}} \]  

(1.66)

\[ \delta^* = \frac{\delta}{3} = \frac{1.826 x}{\sqrt{\text{Re}_x}} \]  

(1.67)

\[ \theta = \frac{2\delta}{15} = \frac{0.730 x}{\sqrt{\text{Re}_x}} \]  

(1.68)

The wall shear stress is then calculated as

\[ \tau_w = 2 \frac{\mu U}{\delta} = 0.365 \frac{\mu U}{x} \sqrt{\text{Re}_x} \]  

(1.69)

The local wall friction coefficient is calculated as

\[ c_f = \frac{\tau_w}{\frac{1}{2} \rho U^2} \]

\[ = \frac{0.365 \frac{\mu U}{x} \sqrt{\text{Re}_x}}{\frac{1}{2} \rho U^2} \]

\[ = 0.730 \frac{\nu}{U_x} \sqrt{\text{Re}_x} \]

\[ = 0.730 \frac{\sqrt{\text{Re}_x}}{\text{Re}_x} \]

\[ = \frac{0.730}{\sqrt{\text{Re}_x}} \]  

(1.70)

The wall friction coefficient is calculated as

\[ c_D = \frac{F_D}{\frac{1}{2} \rho U^2 A} \]  

(1.71)

where
\[ F_D = \int_0^L \tau_w \, dx = \int_0^L 0.365 \frac{\mu U}{x} \sqrt{\text{Re}_x} \, dx \]
\[ = \int_0^l 0.365 \frac{\mu U}{x} \sqrt{\frac{U_x}{v}} \, dx = \int_0^l 0.365 \mu U \sqrt{\frac{U_x}{v}} \frac{1}{2} b \, dx \]  
\[ (1.72a) \]

\[ F_D = b \left( 0.365 \mu U \sqrt{\frac{U}{v}} \right) \left[ \frac{1}{2} \left( \frac{x^2}{1/2} \right) \right]_0^l \]
\[ = b \left( 0.730 \mu U \sqrt{\frac{U}{v}} \right) L^{1/2} = b \left( 0.730 \mu U \sqrt{\text{Re}_L} \right) \]  
\[ (1.72b) \]

Therefore
\[ c_D = \frac{b \left( 0.730 \mu U \sqrt{\text{Re}_L} \right)}{\frac{1}{2} \rho U^2 b L} \]
\[ = 1.46 \frac{\nu}{UL} \sqrt{\text{Re}_L} = 1.46 \sqrt{\frac{\text{Re}_L}{\text{Re}_U}} \]  
\[ (1.73) \]

\[ = 1.46 \sqrt{\frac{1}{\text{Re}_U}} \]

1.4.3 Third order polynomial \[ \frac{u}{U} = \frac{3}{2} \left( \frac{y}{\delta} \right) - \frac{1}{2} \left( \frac{y}{\delta} \right)^3 \]

\[ \tau_w = \rho U^2 \frac{d \theta}{dx} \quad \text{and} \quad \tau_w = \frac{\partial u}{\partial y} \bigg|_{y=0} \]  
\[ (1.74) \]

Since \[ \left. \frac{du}{dy} \right|_{y=0} = \frac{3U}{2\delta} \] therefore
\[ \rho U^2 \frac{d \theta}{dx} = \frac{3 \mu U}{2 \delta} \]  
\[ (1.75) \]

From the previous calculation, we know that \[ \theta = \frac{39 \delta}{280} \], therefore
\[ \frac{39}{280} \rho U^2 \frac{d \delta}{dx} = \frac{3 \mu U}{2 \delta} \]  
\[ (1.76) \]

or
\[ \frac{ax \delta}{20} = \frac{3}{2} \mu U \times \frac{280}{39} \times \frac{dx}{\rho U} = \frac{420}{39} \frac{\mu}{\rho U} dx \] 

(1.77)

We integrate both sides and gives

\[ \frac{\delta^2}{2} = \frac{420}{39} \frac{\mu}{\rho U} x + C \] 

(1.78)

At \( x = 0, \delta = 0 \) and therefore \( C = 0 \)

\[ \frac{\delta^2}{2} = \frac{840 \nu}{39 U^2} x \]

\[ = 21.538 \frac{Ux}{Ux} x \]

\[ = 21.538 \frac{x^2}{Ux/\nu} \]

\[ = 21.538 \frac{x^2}{\text{Re}_x} \]

(1.79)

Then we obtain the following

\[ \delta = \frac{4.64x}{\sqrt{\text{Re}_x}} \] 

(1.80)

The boundary layer displacement and momentum thickness are calculated as follow

\[ \delta^* = \frac{3\delta}{8} \frac{1.74x}{\sqrt{\text{Re}_x}} \] 

(1.81)

\[ \theta = \frac{39\delta}{280} \frac{0.646x}{\sqrt{\text{Re}_x}} \] 

(1.82)

The wall shear stress is then calculated as

\[ \tau_w = \frac{3\mu U}{2\delta} = 0.323 \frac{\mu U}{x \sqrt{\text{Re}_x}} \] 

(1.83)

The local wall friction coefficient is calculated as
The wall friction coefficient is calculated as

\[ c_f = \frac{\tau_w}{1/2 \rho U^2} \]

\[ = \frac{0.323 \mu U}{x} \sqrt{Re_x} \]

\[ = \frac{0.647 U}{U_x} \sqrt{Re_x} \]

\[ = \frac{0.647 \sqrt{Re_x}}{Re_x} \]

\[ = \frac{0.647}{\sqrt{Re_x}} \] (1.84)

The wall friction coefficient is calculated as

\[ c_D = \frac{F_D}{1/2 \rho U^2 A} \] (1.85)

where

\[ F_D = \int_0^L \tau_w bdx = \int_0^L 0.323 \frac{\mu U}{x} \sqrt{Re_x} bdx \]

\[ = \int_0^L 0.323 \frac{\mu U}{x} \sqrt{\frac{U_x}{\nu}} bdx = \int_0^L 0.323 \mu U \sqrt{\frac{U}{\nu}} x^{-3/2} bdx \] (1.86a)

\[ F_D = b \left( 0.323 \mu U \sqrt{\frac{U}{\nu}} \right) \left[ \frac{x^{1/2}}{1/2} \right]_0^L \]

\[ = b \left( 0.646 \mu U \sqrt{\frac{U}{\nu}} \right) L^{1/2} = b \left( 0.646 \mu U \sqrt{L} \right) \] (1.87a)

Therefore

\[ c_D = \frac{b \left( 0.646 \mu U \sqrt{L} \right)}{1/2 \rho U^2 bL} \]

\[ = 1.292 \frac{U}{UL} \sqrt{Re_L} = 1.292 \frac{\sqrt{Re_L}}{Re_L} \] (1.88)
1.4.4 sin-cos velocity distribution: \( \frac{u}{U} = \sin \left( \frac{\pi y}{2 \delta} \right) \)

\[
\tau_w = \rho U^2 \frac{d\theta}{dx} \quad \text{and} \quad \tau_y = \mu \frac{\partial u}{\partial y} \bigg|_{y=0}
\]

Since

\[
\frac{du}{dy} = U \left[ \cos \left( \frac{\pi y}{2 \delta} \right) \right] \times \frac{\pi}{2\delta}
\]

and

\[
\left. \frac{du}{dy} \right|_{y=0} = \frac{U\pi}{2\delta} \quad \text{therefore}
\]

\[
\frac{\mu U\pi}{2\delta} = \rho U^2 \frac{d\theta}{dx}
\]

From the previous calculation, we know that \( \theta = \left( \frac{2}{\pi} - \frac{1}{2} \right) \delta \), therefore

\[
\left( \frac{2}{\pi} - \frac{1}{2} \right) \rho U^2 \frac{d\delta}{dx} = \frac{\mu U\pi}{2\delta}
\]

or

\[
\delta d\delta = \frac{1}{2} \mu U \pi \left( \frac{2\pi}{4 - \pi} \right) \frac{dx}{\rho U^2} = \frac{\pi^2}{(4 - \pi)} \cdot \frac{\mu U}{\rho U^2} dx
\]

or

\[
\delta d\delta = 11.4975 \frac{\mu}{\rho U} dx
\]

Integrating both sides, we get

\[
\frac{\delta^2}{2} = 11.4975 \frac{\mu}{\rho U} x + C
\]

At \( x = 0, \delta = 0 \) and therefore \( C = 0 \)
\[ \delta^2 = 22.995 \frac{u_x}{U} x \]
\[ = 22.995 \frac{u_x}{U} x \]
\[ = 22.995 \frac{x^2}{U_x} \frac{1}{\nu} \]
\[ = 22.995 \frac{x^2}{Re_x} \]  

(1.96)

Then we obtain the following

\[ \delta = \frac{4.795x}{\sqrt{Re_x}} \]  

(1.97)

The boundary layer displacement and momentum thickness are calculated as follow

\[ \delta^* = \left( 1 - \frac{2}{\pi} \right) \delta = \frac{7.847x}{\sqrt{Re_x}} \]  

(1.98)

\[ \theta = \left( \frac{4 - \pi}{2\pi} \right) \delta = \frac{0.651x}{\sqrt{Re_x}} \]  

(1.99)

The wall shear stress is then calculated as

\[ \tau_w = \frac{\mu U \pi}{2\delta} = 0.328 \frac{\mu U}{x} \sqrt{Re_x} \]  

(1.100)

The local wall friction coefficient is calculated as
The wall friction coefficient is calculated as

\[
c_f = \frac{\tau_w}{\frac{1}{2} \rho U^2}
\]

\[
= \frac{0.328 \mu U}{x} \sqrt{Re_x}
\]

\[
= \frac{1}{2} \rho U^2
\]

\[
= 0.655 \frac{U}{U \sqrt{Re_x}} \sqrt{Re_x}
\]

\[
= \frac{0.655}{\sqrt{Re_x}}
\]

The wall friction coefficient is calculated as

\[
c_D = \frac{F_D}{\frac{1}{2} \rho U^2 A}
\]

where

\[
F_D = \int_0^L \tau_w b dx = \int_0^L 0.328 \frac{\mu U}{x} \sqrt{Re_x} b dx
\]

\[
= \int_0^L 0.328 \frac{\mu U}{x} \sqrt{\frac{U}{v}} b dx = \int_0^L 0.328 \mu U \sqrt{\frac{U}{v}} \frac{1}{2} b dx
\]

\[
F_D = b \left( 0.328 \mu U \sqrt{\frac{U}{v}} \right) \left[ \frac{1}{2} \frac{x^2}{L} \right]_0^L
\]

\[
= b \left( 0.656 \mu U \sqrt{\frac{U}{v}} \right) L^{1/2} = b \left( 0.656 \mu U \sqrt{Re_L} \right)
\]

Therefore

\[
c_D = \frac{b \left( 0.656 \mu U \sqrt{Re_L} \right)}{\frac{1}{2} \rho U^2 b L}
\]

\[
= 1.312 \frac{U}{UL} \sqrt{Re_L} = 1.312 \frac{\sqrt{Re_L}}{Re_L}
\]

\[
= \frac{1.312}{\sqrt{Re_L}}
\]
As mentioned earlier, most of the flow is turbulent in nature. When the Reynolds number of the surface exceed approximately $Re = 5 \times 10^5$, the boundary layer transitions from laminar to turbulent.

![Turbulent boundary layer](image)

**Figure 1.7** Turbulent boundary layer

The velocity distribution for the turbulent boundary layer is given by the *one-seventh power law* as follow

$$\frac{u}{U} = \left(\frac{y}{\delta}\right)^{\frac{1}{7}}$$

(1.105)

However, the difficulty arises when determining the wall shear stress as it approaches infinity as below

$$\left.\frac{\partial u}{\partial y}\right|_{y=0} = \frac{1}{7} \frac{U}{\delta^{\frac{1}{7}}} y^{\frac{6}{7}} = \infty.$$  

(1.106)

Here, the correlation for shear stress is taken from experimental work as

$$\tau_w = 0.0226 \rho U^2 \left(\frac{v}{U\delta}\right)^{\frac{1}{4}}$$

(1.107)

### 1.5 Turbulent Boundary Layer

Now we will determine $\delta$, $c_f$ and $c_D$ for the region of turbulent boundary layer.
From the previous calculation, we know that $\theta = \frac{7\delta}{72}$, therefore

$$\frac{7}{72} \rho U^2 \frac{d\delta}{dx} = 0.0226 \rho U^2 \left( \frac{\nu}{U\delta} \right)^{\frac{1}{4}}$$  \hspace{1cm} (1.108)$$

or

$$\delta^{\frac{1}{4}} d\delta = 0.232 \left( \frac{\nu}{U} \right)^{\frac{1}{4}} dx$$  \hspace{1cm} (1.109)$$

We integrate both sides and gives

$$\frac{4}{5} \delta^{5/4} = 0.232 \left( \frac{\nu}{U} \right)^{\frac{1}{4}} x + C$$  \hspace{1cm} (1.110)$$

At $x = 0$, $\delta = 0$ and therefore $C = 0$

$$\delta^{5/4} = 0.291 \left( \frac{\nu}{U} \right)^{\frac{1}{4}} x$$

$$= 0.291 \left( \frac{\nu x}{Ux} \right)^{\frac{1}{4}} x$$

$$= 0.291 \left( \frac{\nu}{U} \right)^{\frac{1}{4}} x^{5/4}$$

$$= 0.291 \frac{x^{5/4}}{Re_x^{\frac{1}{4}}}$$  \hspace{1cm} (1.111)$$

Then we obtain the following

$$\delta = \frac{0.372x}{Re_x^{\frac{1}{5}}}$$  \hspace{1cm} (1.112)$$

The boundary layer displacement and momentum thickness are calculated as follow

$$\delta^* = \frac{\delta}{8} = \frac{0.047x}{Re_x^{\frac{1}{5}}}$$  \hspace{1cm} (1.113)$$

$$\theta = \frac{7\delta}{72} = \frac{0.036x}{Re_x^{\frac{1}{5}}}$$  \hspace{1cm} (1.114)$$

The wall shear stress is then calculated as
\[
\tau_w = 0.0226 \rho U^2 \left( \frac{U}{U \times 0.372x} \right)^{\frac{1}{4}} \\
= 0.0289 \rho U^2 \left( \frac{\text{Re}_{x}^{\frac{1}{5}}}{\text{Re}_{x}} \right)^{\frac{1}{4}} = 0.0289 \rho U^2 \left( \frac{1}{\text{Re}_{x}^{\frac{4}{5}}} \right)^{\frac{1}{4}} \\
= \frac{0.0289 \rho U^2}{\text{Re}_{x}^{\frac{1}{5}}} 
\]

The local wall friction coefficient is calculated as

\[
c_f = \frac{\tau_w}{\frac{1}{2} \rho U^2} \\
= \frac{0.0289 \rho U^2 \left( \frac{1}{\text{Re}_{x}^{\frac{4}{5}}} \right)^{\frac{1}{4}}}{\frac{1}{2} \rho U^2} \\
= 0.0579 \left( \frac{1}{\text{Re}_{x}^{\frac{4}{5}}} \right)^{\frac{1}{4}} \\
= \frac{0.0579}{\text{Re}_{x}^{\frac{1}{5}}} 
\]

The wall friction coefficient is calculated as

\[
c_D = \frac{F_D}{\frac{1}{2} \rho U^2 A} 
\]

where
\[ F_D = \int_0^L x \, \kappa b \, dx = \int_0^L \frac{0.0289 \rho U^2}{Re_x^{1/5}} \, b \, dx \]
\[ = \int_0^L 0.0289 \rho U^2 \left( \frac{\nu}{U x} \right)^{1/5} b \, dx = \int_0^L 0.0289 \rho U^2 b \left( \frac{\nu}{U} \right)^{1/5} x^{-2} \, dx \]
\[ F_D = 0.0289 \rho U^2 b \left( \frac{\nu}{U} \right)^{1/5} \left[ x^{4/5} \right]_0^L \]
\[ = 0.036 \rho U^2 b \left( \frac{\nu}{U} \right)^{1/5} L^{4/5} \]  
\[ (1.118a) \]
\[ (1.118b) \]

Therefore

\[ c_D = \frac{0.036 \rho U^2 b \left( \frac{\nu}{U} \right)^{1/5} L^{4/5}}{\frac{1}{2} \rho U^2 b L} \]
\[ = 0.072 \left( \frac{\nu}{U} \right)^{1/5} \frac{1}{L^{4/5}} = 0.072 \left( \frac{1}{UL} \right)^{1/5} \]  
\[ = \frac{0.072}{Re_L^{1/5}} \]  
\[ (1.119) \]

Now the formulated expression for \( \delta \), \( c_f \) and \( c_D \) for various velocity distribution in laminar region and turbulent region are tabulated in Table 1.3

<table>
<thead>
<tr>
<th>Types of velocity distribution</th>
<th>( \delta )</th>
<th>( c_f )</th>
<th>( c_D )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Blassius Solution</td>
<td>( \frac{5x}{\sqrt{Re_x}} )</td>
<td>( \frac{0.664}{\sqrt{Re_x}} )</td>
<td>( \frac{1.328}{\sqrt{Re_L}} )</td>
</tr>
<tr>
<td>Linear profile, ( \frac{u}{U} = \frac{3y}{\delta} )</td>
<td>( \frac{3.46x}{\sqrt{Re_x}} )</td>
<td>( \frac{0.58}{\sqrt{Re_x}} )</td>
<td>( \frac{1.15}{\sqrt{Re_L}} )</td>
</tr>
</tbody>
</table>

Table 1.3 Results for various assumed laminar and turbulent flow velocity profiles
Parabolic profile,
\[
\frac{u}{U} = \left( \frac{y}{\delta} \right)^2 + 2 \left( \frac{y}{\delta} \right)
\]
\[
\frac{5.48x}{\sqrt{Re_x}} \quad \frac{0.730}{\sqrt{Re_x}} \quad \frac{1.46}{\sqrt{Re_L}}
\]

Cubic profile,
\[
\frac{u}{U} = \frac{3}{2} \left( \frac{y}{\delta} \right) - \frac{1}{2} \left( \frac{y}{\delta} \right)^3
\]
\[
\frac{4.64x}{\sqrt{Re_x}} \quad \frac{0.647}{\sqrt{Re_x}} \quad \frac{1.292}{\sqrt{Re_L}}
\]

Sin-Cos profile,
\[
\frac{u}{U} = \sin \left( \frac{\pi y}{2\delta} \right)
\]
\[
\frac{4.795x}{\sqrt{Re_x}} \quad \frac{0.655}{\sqrt{Re_x}} \quad \frac{1.312}{\sqrt{Re_L}}
\]

Turbulent profile,
\[
\frac{u}{U} = \left( \frac{y}{\delta} \right)^{1/5}
\]
\[
\frac{0.372x}{Re_x^{1/5}} \quad \frac{0.058}{Re_x^{1/5}} \quad \frac{0.072}{Re_L^{1/5}}
\]

Question:

1. Consider a flow over a horizontal flat plate (1.25m x 2.5m) with velocity 3.0m/s. Calculate
   a. Boundary layer thickness at trailing edge
   b. Shear stress at the middle of the flat plate
   c. Resultant drag force on both sides of the flat plate
   
   (Take \( \rho = 850 \text{ kg/m}^3 \), \( v = 10^{-5} \text{ m}^2/\text{s} \))

Solution

Given: \( U = 3.0 \text{ m/s} \), \( b = 1.25 \text{ m} \), \( L = 2.5 \text{ m} \)

\[
\text{Re}_L = \frac{UL}{\nu} = \frac{3.0 \times 2.5}{10^{-5}} = 7.5 \times 10^5
\]

Since \( \text{Re}_L > 5 \times 10^5 \) so the boundary layer at the trailing edge is turbulent. The Von-Karmann equation gives,
\[
\delta_x = \frac{0.372x}{\text{Re}^{1/5}}
\]

At the trailing edge \( x = L \)

\[
\delta_L = \frac{0.372 \times 2.5}{(7.5 \times 10^5)^{1/5}} = 0.06215 \text{ m} = 62.15 \text{ mm (Ans)}
\]

**Shear stress at the middle of the flat plate**

At the middle of the plate, \( x = \frac{L}{2} = 1.25 \text{ m} \),

the Reynolds number is calculated as

\[
\text{Re}_x = \frac{Ux}{\nu} = \frac{3.0 \times 1.25}{10^{-5}} = 3.75 \times 10^5 \text{ m}.
\]

Hence the boundary layer at this point is laminar and the Blassius equation for the local wall friction coefficient gives

\[
C_f = \frac{664}{\text{Re}^{1/2}} = \frac{0.664}{(3.75 \times 10^5)^{1/2}} = 0.00108
\]

By definition \( \tau_w = C_f \times \frac{\rho U^2}{2} = 0.00108 \times \frac{850 \times 3.0^2}{2} = 4.147 \text{ N/m}^2 \text{ (Ans)} \)

**Resultant drag force on both sides of the flat plate, } F_D \)

\[
F_D = C_D \times \frac{1}{2} \rho A U^2
\]

where \( C_D = \frac{0.072}{\text{Re}_L^{1/5}} = \frac{0.072}{(7.5 \times 10^5)^{1/5}} = 0.00481 \)

and \( A = \text{area of the plate} = L \times b = 2.5 \times 1.25 = 3.125 \text{ m}^2 \).

\[
F_D = 0.00481 \times \frac{1}{2} \times 850 \times 3.125 \times 3.0^2 = 57.494 \text{ N}
\]

The resultant force on both sides of the plate is calculated as
\[ \therefore F_D \times 2 \text{ sides} = 114.988 \text{ N (Ans)} \]

2. A submarine can be assumed to have cylindrical shape with rounded nose. Assuming its length to be 50m and diameter 5.0m. Determine the total power required to overcome boundary friction if it cruises at 8m/s velocity in sea water.

(Take \( \rho = 1030 \text{ kg/m}^3 \), \( \nu = 10^{-6} \text{ m}^2/\text{s} \))

**Solution**

Given: \( U = 8.0 \text{ m/s}, \ D = 5.0 \text{ m}, \ L = 50 \text{ m} \)

The transition of laminar to turbulent boundary layer occurs at \( \text{Re} = 5 \times 10^5 \) or

\[
\frac{Ux}{\nu} = \frac{8 \times x}{10^{-6}} = 5 \times 10^5 \text{ at}
\]

\( x = 0.0652 \text{ m} \)

Since the laminar boundary layer is too small and can we can assume the drag force is due to the turbulent boundary layer only. Hence, \( C_D \) is given by

\[
C_D = \frac{0.072}{\text{Re}^{0.15}} = \frac{0.072}{\left( \frac{8 \times 50}{10^{-6}} \right)^{0.15}} = 0.00137
\]

Area, \( A = \pi DL = \pi \times 5 \times 50 = 785.4 \text{ m}^2 \)

\[ \therefore \text{Drag force} \ F_D = C_D \times \frac{1}{2} \rho AU^2 = 0.00137 \times \frac{1}{2} \times 1030 \times 785.4 \times 8^2 = 35464 \text{ N} \]

Hence total power required to overcome boundary friction,

\[ P = F_D U = 35464 \times 8 = 283712 \text{ Watt or 283.7 kW (Ans)} \]
3. A barge with a rectangular bottom surface 30m long times 10m wide is traveling down a river with a velocity of 0.6m/s. A laminar boundary layer exists up to a Reynolds number equivalent to $5 \times 10^5$ and subsequently abrupt transition occurs to turbulent boundary layer.

a. The maximum distance from the leading edge up to which laminar boundary layer thickness persists and the boundary layer thickness at that point.

b. The total drag force on the flat bottom surface of the barge.

c. The power required to push the bottom surface through water at the given velocity.

(Take $\rho = 998 \text{ kg/m}^3, \nu = 10^{-6} \text{ m}^2/\text{s}$)

**Solution**

Given: $U = 0.6 \text{ m/s}, b = 10.0 \text{ m}, L = 30 \text{ m}$

**Maximum distance of laminar boundary layer**

Maximum distance of laminar boundary layer occurs at the point of transition from laminar to turbulent or at $\text{Re}_x = 5 \times 10^5 = \frac{UL}{\nu} = \frac{0.6 \times x}{10^{-6}}$ and therefore

$x = 0.833 \text{ m (Ans)}$

At this point, the boundary layer thickness can be calculated as

$$\delta = \frac{5x}{\sqrt{\text{Re}_x}} = \frac{5 \times 0.833}{\sqrt{(5 \times 10^5)^{1/2}}} = 0.00589 \text{ m or 5.89 mm (Ans)}$$

**The total drag force on the flat bottom surface of the barge**

Since the length of laminar boundary layer is very small, contribution to total drag from laminar boundary layer is negligible.

The Reynolds number at the trailing edge, $\text{Re}_L = \frac{UL}{\nu} = \frac{0.6 \times 30}{10^{-6}} = 1.8 \times 10^7$
The wall friction coefficient is calculated as 
\[ C_D = \frac{0.072}{\text{Re}^{1/5}} = \frac{0.072}{\left(1.8 \times 10^7\right)^{1/5}} = 0.00255 \]

Area, \( A = bL = 10 \times 30 = 300 \text{ m}^2 \)

\[ \therefore \text{Drag force } F_D = C_D \times \frac{1}{2} \rho A U^2 = 0.00255 \times \frac{1}{2} \times 998 \times 300 \times 0.6^2 = 137.4 \text{ N} \]

\[ = 137.4 \text{ N (Ans)} \]

The power required to push the bottom surface through water at the given velocity can be calculated as follow

\[ P = F_D U = 137.4 \times 0.6 = 82.45 \text{ Watt (Ans)} \]

4. Determine the total power required to pull the timber (diameter 0.4m and length 15m) with velocity 0.8m/s. Only half of the timber on the water surface. Neglect the drag force due to front surface.

Take \( C_{D_{\text{turbulent}}} = \frac{1.31}{\sqrt{\text{Re}}} \), \( C_{D_{\text{laminar}}} = \frac{0.074}{5\sqrt{\text{Re}}} \), \( \rho_{\text{water}} = 1000 \text{ kg/m}^3 \), \( v = 1 \times 10^{-6} \text{ m}^2/\text{s} \)

Solution

Given: \( U = 0.8 \text{ m/s}, \ D = 0.4 \text{ m}, \ L = 15 \text{ m} \)

The transition of laminar to turbulent boundary layer occurs at \( \text{Re} = 5 \times 10^5 \) or

\[ \frac{Ux}{\nu} = \frac{0.8 \times x}{10^{-6}} = 5 \times 10^5 \text{ at} \]

\[ x = 0.652 \text{ m} \]

Since the laminar boundary layer is too small and can we can assume the drag force is due to the turbulent boundary layer only. Hence, \( C_D \) is given by

\[ C_D = \frac{0.072}{\text{Re}^{1/5}} = \frac{0.072}{\left(0.8 \times 15 \times 10^{-6}\right)^{1/5}} = 0.00276 \]

Area, \( A = \frac{\pi D}{2} L = \frac{\pi \times 0.4}{2} \times 15 = 9.426 \text{ m}^2 \)
\[ \text{Drag force } F_D = C_D \times \frac{1}{2} \rho AU^2 = 0.00276 \times \frac{1}{2} \times 1000 \times 9.426 \times 0.8^2 = 8.325 \text{ N} \]

Hence total power required to overcome boundary friction,

\[ P = F_D U = 8.325 \times 0.8 = 6.66 \text{ Watt (Ans)} \]

5. Calculate the ratio of drag force on the front half and rear half of the flat plate (Length, \( L \) and width, \( b \)) in a uniform velocity, if the boundary layer is turbulent over the whole plate.

**Solution**

The wall friction coefficient \( C_D \) for turbulent boundary layer is given by

\[ C_D = \frac{0.0072}{\text{Re}_{L/2}^{1/8}} \]

Reynolds number for the first half of the plate, \( \text{Re}_{L/2} = \frac{UL}{2} \nu \)

Drag force for the first half of the plate is calculated as

\[ F_D = C_D \times \frac{1}{2} \rho bLU^2 = \frac{0.0072}{\left(\frac{UL}{2\nu}\right)^{1/8}} \times \frac{1}{2} \times \rho \times b \frac{L}{2} \times U^2 \]

\[ = \left[ \frac{0.0072}{\left(\frac{UL}{\nu}\right)^{1/8}} \times \frac{1}{2} \times \rho \times bL \times U^2 \right] \left(\frac{2^{1/8}}{2}\right) \]

Similarly, the drag force for the entire plate is calculated as

\[ F_D = C_D \times \frac{1}{2} \rho bLU^2 = \frac{0.0072}{\left(\frac{UL}{\nu}\right)^{1/8}} \times \frac{1}{2} \times \rho \times bL \times U^2 \]

Then the drag force for the rear half of the plate can be calculated as follow

Drag force for the entire plate – drag force for the front half of the plate
or

\[
\begin{bmatrix}
\frac{0.0072}{(UL)^{\frac{1}{5}}} \times \frac{1}{2} \times \rho \times bL \times U^2 \\
\frac{0.0072}{(UL)^{\frac{1}{5}}} \times \frac{1}{2} \times \rho \times b \times L \times U^2
\end{bmatrix}
- \begin{bmatrix}
\frac{0.0072}{(UL)^{\frac{1}{5}}} \times \frac{1}{2} \times \rho \times b \frac{L}{2} \times U^2
\end{bmatrix} =
\end{equation}

\[
\begin{bmatrix}
\frac{0.0072}{(UL)^{\frac{1}{5}}} \times \frac{1}{2} \times \rho \times bL \times U^2 \\
\frac{0.0072}{(UL)^{\frac{1}{5}}} \times \frac{1}{2} \times \rho \times bL \times U^2 \left(1 - \frac{2^{\frac{1}{5}}}{2}\right)
\end{bmatrix}
= \frac{0.5743}{1 - 0.5743}
= 1.374 \text{ (Ans)}
\]